Distributed 2-approximation algorithm for Vertex Cover

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Weighted Vertex Cover

Given a node-weighted graph, find a minimum-weight subset of the nodes, touching all the edges.
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A sequential 2-approximation algorithm

- “Edge discount” ---
  reduce edge endpoints’ costs equally.
- Do edge discounts until zero-cost nodes form a cover.
  Return the cover formed by the zero-cost nodes.

[Bar-Yehuda and Even, 1981]
edge discount operation

Reduce both endpoints’ costs equally.
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**Theorem:** \( \text{cost}(C) \) is at most twice the minimum possible cost.

**Proof:**

(i) \( \text{cost}(C) \) is at most twice the sum of the discounts.

(ii) The sum of the discounts is at most the optimal cost.
proof:

\[ \sum_{v \in C} \text{cost}(v) = \sum_{v \in C} \sum_{e \sim v} \text{discount}(e) \leq \sum_{e \in E} 2 \text{discount}(e) \]
step (ii): \textit{Optimal cost is at least the sum of the discounts.}

\textbf{proof:}

\[
\sum_{v \in \text{OPT}} \text{cost}(v) \geq \sum_{v \in \text{OPT}} \sum_{e \sim v} \text{discount}(e) \geq \sum_{e \in E} \text{discount}(e)
\]
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next: fast distributed implementation of BY&E algorithm
each node knows only its neighbors
distributed computation

- Proceed in rounds.
- In each round:
  
  Each node exchanges $O(1)$ messages with immediate neighbors,

  then does some computation.

**goal:** Finish in a small (logarithmic) number of rounds.
Each round:

1. form independent rooted “stars”

2. coordinate discounts within stars

Done when zero-cost vertices cover all edges.

_goal:_ Done after $O(\log n)$ rounds ($n = \#nodes$).
how to form stars

1. Each node randomly chooses to be “boy” or “girl” (just for this round).

2. For the round, use only edges from boys to higher-cost (or equal-cost) girls. (Pretend other edges don’t exist.)†

3. Each boy chooses a random neighbor (girl of ≥ cost).

† In each round, every edge has a one in four chance of being used. (...) will be used if low-cost endpoint is boy, high-cost endpoint is girl)
How to form stars:

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how girls allocate discounts

- Each girl allocates discounts greedily, in alphabetic order.
- If she *partially allocates* some boy’s discount, then...

  with probability 1/2:
  1. She revokes discounts to all other boys.
  2. She allocates full discount to that boy.

- Some boys may be *jilted* (have no chance for discount).

Tuesday, February 5, 13
Each girl allocates discounts greedily, in alphabetical order.

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---

**Diagram:**

- **8** (jilted)
- **4** (full discount)
- **3** (partial discount)
- **1**
- **0**
- **0**
- **0**
- **0**
- **2**
- **1**
- **3**

50%
Each girl allocates discounts greedily, in alphabetical order.

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Diagram:

- 8
  - 0
  - 0
  - 7
  - 0
  - 0
  - 0
  - 0

- 4
  - 0

- 3
  - 3

- 7
  - 7

- 1
  - 1

- 2
  - 2

- 3
  - 3

---

50% not jilted

50% still jilted
Each girl allocates discounts greedily, in alphabetical order.

If she *partially allocates* some boy’s discount, then...

with probability $1/2$:

1. She revokes discounts to all other boys.
2. She allocates full discount to that boy.

---

\[ \begin{array}{c}
\text{not jilted} \\
\text{--- full discount} \\
\text{still jilted} \\
\end{array} \]
each of girl’s boys is either:

- **jilted** (girl gives no chance of any discount)
- **not jilted** (girl gives at least 50% chance of discount)

```plaintext
definitely jilted.
no chance of any discount.
```
each of girl’s boys is either:

- **jilted** (girl gives no chance of any discount)
- **not jilted** (girl gives at least 50% chance of discount)

\[8\]

- 4 (50%)
- 3 (50%)
- 7 (50%)

- 0
- 0
- 0
- 1
- 2
- 3

\[\{\]

**not jilted.**
50% chance of full discount.

**definitely jilted.**
no chance of any discount.
each of girl’s boys is either:

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- **not jilted.**
  - 50% chance of full discount.

- **definitely jilted.**
  - no chance of any discount.
recap of algorithm

in each round:

1. Each node randomly chooses to be “boy” or “girl”. Only edges from boys to higher-cost girls are used.

2. To form stars, each boy chooses a random neighbor (girl).

3. To allocate discounts within stars:
   (a) Each girl allocates greedily in alphabetical order.
   (b) If a boy is partially allocated, with probability $1/2$, she gives a full discount to just that boy.
analysis

- Guaranteed to return a 2-approximate solution, since it implements the edge-discount algorithm.

- What about running time?

  **Goal:** Show $O(\log n)$ rounds (w.h.p.).
analyis of number of rounds

 lemma: In each round, in expectation,
a constant fraction of each boy’s active edges are deleted.

 proof: (next)

corollary: Number of rounds is $O(\log n^2) = O(\log n)$
in expectation and with high probability.
Lemma: In each round, in expectation, a constant fraction of each boy's active edges are deleted.

Proof: Fix any boy. For the analysis, condition on the random choices of all other boys. (Imagine that the boy chooses his girl after every other boy chooses.)

- For each girl neighbor, what would happen if he were to choose that girl?
**lemma:** In each round, in expectation, a constant fraction of each boy’s active edges are deleted.

**proof:** Fix any boy. For the analysis, condition on the random choices of all other boys. (Imagine that the boy chooses his girl after every other boy chooses.)

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(Imagine that the boy chooses his girl *after* every other boy chooses.)

- For each girl neighbor, what would happen if he were to choose *that* girl?
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**Lemma:** In each round, in expectation, a constant fraction of each boy’s active edges are deleted.

**Proof:**

Fix any boy. For the analysis, condition on the random choices of all other boys. (Imagine that the boy chooses his girl after every other boy chooses.)

- For each girl neighbor, what would happen if he were to choose that girl?

**Key Observation:**

- This girl would not jilt boy. Would have 50% chance of zeroing boy’s cost.
- This girl would jilt boy. Girl’s own cost will definitely go to zero, even if boy doesn’t choose girl.
- This girl would not jilt boy. Would reduce boy’s cost to zero.
- This girl would not jilt boy. Would have 50% chance of zeroing boy’s cost.
- This girl would jilt boy. Girl’s own cost will definitely go to zero, even if boy doesn’t choose girl.

**Key Observation:**

- Girl would jilt boy $\Rightarrow$ her cost is going to zero regardless of what boy does.
- Girl would not jilt boy $\Rightarrow$ if boy chooses her, she has at least a 50% chance of zeroing boy’s cost...
key observation:
girl **would jilt** boy $\Rightarrow$ her cost is going to zero regardless of what boy does.
girl **would not jilt** boy $\Rightarrow$ if boy chooses her, she has at least a 50% chance of zeroing boy's cost...

case (i): At least half of boy's girls would jilt him.
  $\Rightarrow$ At least half of boy's edges will be deleted regardless of what boy does.

case (ii): At least half of boy's girls would not jilt him.
  $\Rightarrow$ Boy has at least a 50% chance of choosing a girl who has at least a 50% chance of zeroing his cost (deleting all his edges).
thank you

• deterministic $O(\log^c n)$-round algorithm?