On the Number of Iterations for Dantzig-Wolfe Optimization and Packing-Covering Approximation Algorithms

Phil Klein
Brown University

Neal Young
Dartmouth College

simple multicommodity flow problem

$P = \{s_i \rightarrow t_i \text{ paths}\}$.

Route at least 1 total unit of flow,

respecting capacity constraint c ($0 < c < 1$).

simple multicommodity flow algorithm

$P = \{s_i \rightarrow t_i \text{ paths}\}$.

Repeat for T iterations:

Route $1/T$ units of flow on min-cost path in P,
where cost of edge e is $(1+\epsilon)^{T\,\mathsf{flow}(e)}$.

performance guarantee

THM: If flow of congestion c exists, then algorithm returns flow of congestion $c(1+\epsilon)$

provided $T \geq 3\,\dfrac{\ln(m)}{c\,\epsilon^2}$.

generic packing problem

input: real matrix A, vector b, generic polytope P

output: $x \in P$ such that $Ax \leq b$.

generic packing problem

input: real matrix A, vector b, generic polytope P

oracle for P: given vector c returns $\arg\min_{x \in P} c \cdot x$.

ϵ-approximate solution: $x \in P$ such that $Ax \leq (1+\epsilon)b$.

Lagrangian Relaxation

idea: replace constraints by costs

1950: Ford, Fulkerson
Reduced multicommod. flow to iterated min-cost flow.
1960: Dantzig, Wolfe
Generalized to generic packing problem.
1970: Held, Karp
Reduced TSP l.b. to iterated min. 1-spanning-tree.
1990: Shahrokhi, Matula
Multicommodity flow, guaranteed convergence rate.
⋮

1995: Plotkin, Shmoys, Tardos
Generalized to generic packing problem.

Iterations prop. to $\rho \ln(m)/\epsilon^2$

$\rho =$ "width", $m =$ # constraints.

⋮

main result: a lower bound

THM: Any ϵ-approximation algorithm for the generic packing problem requires a number of iterations prop. to $T = \rho \ln(m)/\epsilon^2$ for sufficiently large m.

proof idea ($\rho = 2$):

Reduce to question about two-player zero-sum matrix games.

value(M) $= \min_{x \in P} \max_j (M\,x)_j$, where $P = \{x \geq 0 : \sum_i x_i = 1\}$.

THM: Let M be a random matrix in $\{0,1\}^{m \times \sqrt{m}}$.
With high probability, every $m \times T$ submatrix B of M has

$$\text{value}(B) > (1+\epsilon)\text{value}(M)$$

where $T = \Omega(\ln(m)/\epsilon^2)$.

COROLLARY: At least T oracle calls to know value(M) within $1+\epsilon$.

underlying idea:

Show $m \times T$ submatrix has high value with high probability:

Discrepancy theory.