

# K-Medians, Facility Location, and the Chernoff-Wald Bound

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## Abstract

We study the general (non-metric) facility-location and weighted  $k$ -medians problems, as well as the fractional facility-location and  $k$ -medians problems. We describe a natural randomized rounding scheme and use it to derive approximation algorithms for all of these problems.

For facility location and weighted  $k$ -medians, the respective algorithms are polynomial-time  $[H_\Delta k + d]$ - and  $[(1 + \epsilon)d; \ln(n + n/\epsilon)k]$ -approximation algorithms. These performance guarantees improve on the best previous performance guarantees, due respectively to Hochbaum (1982) and Lin and Vitter (1992). For fractional  $k$ -medians, the algorithm is a new, Lagrangian-relaxation,  $[(1 + \epsilon)d, (1 + \epsilon)k]$ -approximation algorithm. It runs in  $O(k \ln(n/\epsilon)/\epsilon^2)$  linear-time iterations.

For fractional facilities-location (a generalization of fractional weighted set cover), the algorithm is a Lagrangian-relaxation,  $[(1 + \epsilon)k]$ -approximation algorithm. It runs in  $O(n \ln(n)/\epsilon^2)$  linear-time iterations and is essentially the same as an unpublished Lagrangian-relaxation algorithm due to Garg (1998). By recasting his analysis probabilistically and abstracting it, we obtain an interesting (and as far as we know new) probabilistic bound that may be of independent interest. We call it the *Chernoff-Wald* bound.

## 1 Problem definitions

The input to the **weighted set cover problem** is a collection of sets, where each set  $s$  is given a cost  $\text{cost}(s) \in \mathbb{R}_+$ . The goal is to choose a cover (a collection of sets containing all elements) of minimum total cost.

The (un capacitated) **facility-location problem** is a generalization of weighted set cover in which each set  $f$  (called a “facility”) and element  $c$  (called a “customer”) are given a distance  $\text{dist}(f, c) \in \mathbb{R}_+ \cup \{\infty\}$ . The goal is to choose a set of facilities  $F$  minimizing  $\text{cost}(F) + \text{dist}(F)$ , where  $\text{cost}(F)$ , the *facility cost of  $F$* , is  $\sum_{f \in F} \text{cost}(f)$  and  $\text{dist}(F)$ , the *assignment cost of  $F$* , is  $\sum_c \min_{f \in F} \text{dist}(f, c)$ .

Fig. 1 shows the standard integer programming formulation of the problem — the *facility-location IP* [12, p. 8]. The facility-location *linear program (LP)* is the same except without the constraint “ $x(f) \in \{0, 1\}$ ”. A *fractional solution* is a feasible solution to the LP. *Fractional facility location* is the problem of solving the LP.

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$$\begin{array}{l} \text{minimize}_x \quad d + k \\ \text{subject to} \quad \left\{ \begin{array}{ll} \text{cost}(x) \leq k & \\ \text{dist}(x) \leq d & \\ \sum_f x(f, c) = 1 & (\forall c) \\ x(f, c) \leq x(f) & (\forall f, c) \\ x(f, c) \geq 0 & (\forall f, c) \\ x(f) \in \{0, 1\} & (\forall f) \end{array} \right. \end{array}$$

Figure 1: The facility-location IP. Above  $d$ ,  $k$ ,  $x(f)$  and  $x(f, c)$  are variables,  $\text{dist}(x)$ , the *assignment cost of  $x$* , is  $\sum_{f,c} x(f, c) \text{dist}(f, c)$ , and  $\text{cost}(x)$ , the *facility cost of  $x$* , is  $\sum_f x(f) \text{cost}(f)$ .

The **weighted  $k$ -medians problem** is the same as the facility location problem except for the following: a positive real number  $k$  is given as input, and the goal is to choose a subset  $F$  of facilities minimizing  $\text{dist}(F)$  subject to the constraint  $\text{cost}(F) \leq k$ . The standard integer programming formulation is the  *$k$ -medians IP*, which differs from the IP in Fig. 1 only in that  $k$  is given, not a variable, and the objective function is  $d$  instead of  $d + k$ . The (*unweighted*)  $k$ -medians problem is the special case when each  $\text{cost}(f) = 1$ . For the *fractional  $k$ -medians problem*, the input is the same; the goal is to solve the linear program obtained by removing the constraint “ $x(f) \in \{0, 1\}$ ” from the  $k$ -medians IP.

We take the “size” of each of the above problems to be the number of pairs  $(f, c)$  such that  $\text{dist}(f, c) < \infty$ . We assume the size is at least the number of customers and facilities.

By an  $[\alpha(d); \beta(k)]$ -**approximation algorithm** for  $k$ -medians, we mean an algorithm that, given a problem instance for which there exists a fractional solution of assignment cost  $d$  and facility cost  $k$ , produces a solution of assignment cost at most  $\alpha(d)$  and facility cost at most  $\beta(k)$ . We use similar non-standard notations for facility location, set cover, and  $k$ -set cover. For instance, by a  $[d + 2k]$ -approximation algorithm for facility location, we mean an algorithm that, given an instance for which there exists a fractional solution of assignment cost  $d$  and facility cost  $k$ , produces a solution for which the assignment cost plus the facility cost is at most  $d + 2k$ .

## 2 Background

In the mid 1970's Johnson and Lovasz gave a greedy  $[H_\Delta k]$ -approximation algorithm for unweighted set cover [9, 11]. In 1979 Chvatal generalized it to a  $[H_\Delta k]$ -approximation algorithm for weighted set cover [3].

In 1982 Hochbaum gave a greedy  $[H_\Delta(d+k)]$ -approximation algorithm for the uncapacitated facility-location problem by an implicit reduction to the weighted set-cover problem [8].<sup>1</sup> Above  $\Delta$  is at most the maximum, over all facilities  $f$ , of the number of customers  $c$  such that  $\text{dist}(f, c) < \infty$ .

In 1992 Lin and Vitter gave a polynomial-time  $[(1+\epsilon)d; (1+1/\epsilon)(\ln n + 1)k]$ -approximation algorithm for the  $k$ -medians problem [10]. (Here  $\epsilon > 0$  is an input parameter that tunes the tradeoff between the two criteria and  $n$  is the number of customers.) Their algorithm combines a greedy algorithm with a technique they call *filtering*.

In 1994 Plotkin, Shmoys, and Tardos (PST) gave Lagrangian-relaxation algorithms for general packing and covering problems [13]. As a special case, their algorithms imply a  $[(1+\epsilon)k]$ -approximation algorithm for fractional set cover that runs in  $O(k \ln(n)/\epsilon^2)$  linear-time iterations. Here  $n$  is the number of elements.

In 1998 Garg generalized and simplified the PST set cover algorithm to obtain a  $[(1+\epsilon)k]$ -approximation algorithm for fractional *weighted* set cover [6]. Garg's algorithm runs in  $O(n \ln(n)/\epsilon^2)$  linear-time iterations. By Hochbaum's reduction, one can use Garg's algorithm as a  $[(1+\epsilon)k]$ -approximation algorithm for fractional *facility location*. The running time is the same, where  $n$  is the number of customers.

Recent work has focused on *metric*  $k$ -medians and facility location problems. In the metric versions, the distance function is assumed to satisfy the triangle inequality. For example,  $[O(d+k)]$ -approximation algorithms have recently been shown for the metric facility-location problem [7, 15]. Charikar, Guha, Tardos and Shmoys [2] recently gave an  $[O(d); k]$ -approximation algorithm for metric  $k$ -medians. Many of these algorithm first solve the fractional problems and then round the fractional solutions.

<sup>1</sup>Hochbaum's reduction is easy to adapt in order to reduce unweighted  $k$ -medians to a variant of set cover that we call  $k$ -set cover. The reduction also extends naturally to the fractional problems. (See appendix for details.) This shows a loose equivalence between facility location and weighted set cover, and between  $k$ -medians and  $k$ -set cover. However, Hochbaum's reduction does not preserve the distinction between facility costs and assignment costs. For this reason, we work with the facility location and  $k$ -medians representations. The algorithms and analyses we give for facility location easily imply corresponding results for weighted set cover, and the results for  $k$ -medians are straightforward to adapt to  $k$ -set cover.

## 3 Results

Fig. 2 shows the simple randomized rounding scheme at the center of all our results. With minor variations (e.g. Fig. 4), this rounding scheme can be used as the basis for approximation algorithms for set cover, weighted set cover, facility location, and  $k$ -medians, and as the basis for Lagrangian-relaxation algorithms for the fractional variants of these problems.

Although essentially the same rounding scheme suffices for each of these problems, the respective probabilistic *analyses* require different (albeit standard) techniques in each case. For set cover, a simple direct analysis suffices [16]. For *weighted* set cover, facility location, and  $k$ -medians, a basic probabilistic lemma called *Wald's inequality* is necessary. For *fractional* set cover and  $k$ -medians, the analysis rests on the *Chernoff* bound. For fractional *weighted* set cover and facility location, the analysis is a simple application of what we call the *Chernoff-Wald* bound.

For each problem, we apply the method of conditional probabilities to the rounding scheme in order to derive a corresponding approximation algorithm. The structure of each resulting algorithm, being closely tied to the underlying probabilistic analysis, ends up differing substantially from problem to problem.

For facility location, the resulting algorithm (Fig. 2) is a randomized rounding, polynomial-time  $[d + H_\Delta k]$ -approximation algorithm. The performance guarantee improves over Hochbaum's algorithm with respect to the assignment costs.

For weighted  $k$ -medians, the resulting algorithm (Fig. 3) is a greedy  $[(1+\epsilon)d; \ln(n+n/\epsilon)k]$ -approximation algorithm. In comparison to Lin and Vitter's algorithm, the performance ratio with respect to the facility costs is better by a factor of roughly  $1/\epsilon$ .

For *fractional*  $k$ -medians, the algorithm (Fig. 5) is a  $[(1+\epsilon)d, (1+\epsilon)k]$ -approximation algorithm. It is a Lagrangian-relaxation algorithm and runs in  $O(k \ln(n/\epsilon)/\epsilon^2)$  linear-time iterations. This is a factor of  $n$  faster than the best bound we can show by applying the general algorithm of Plotkin, Shmoys and Tardos.

Finally, for fractional facility location, the algorithm (see Fig. 6) is a  $[(1+\epsilon)(d+k)]$ -approximation, Lagrangian-relaxation algorithm. It runs in at most  $O(n \ln(n)/\epsilon^2)$  iterations, where each iteration requires time linear in the input size times  $\ln n$ . This algorithm is the same as the unpublished fractional weighted set-cover algorithm due to Garg [6].

The main interest of this last result is not that we improve Garg's algorithm (we don't!), but that we recast and abstract Garg's analysis to obtain an apparently new (?) probabilistic bound — we call it the Chernoff-Wald bound — that may be of general interest

for *probabilistic* applications.<sup>2</sup>

A basic contribution of this work is to identify and abstract out (using the probabilistic method) common techniques underlying the design and analysis of Lagrangian-relaxation and greedy approximation algorithms.

#### 4 Wald's inequality and Chernoff-Wald bound

Before we state and prove Wald's inequality and the Chernoff-Wald bound, we give some simple examples. Suppose we perform repeated trials of a random experiment in which we roll a 6-sided die and flip a fair coin. We stop as soon as the total of the numbers rolled exceeds 3494. Let  $T$  be the number of trials. Let  $D \leq 3500$  be the total of the numbers rolled. Let  $H$  be the number of flips that come up heads. Since the expectation of the number in each roll is 3.5, Wald's implies  $E[D] = 3.5E[T]$ , which implies  $E[T] \leq 1000$ . Since the probability of a head in each coin flip is 0.5, Wald's implies  $E[H] = 0.5E[T]$ . Thus we can conclude  $E[H] \leq 500$ .

Now modify the experiment so that in each trial, each person *in a group of 50* flips their own fair coin. Let  $M$  be the maximum number of heads any person gets. Then Chernoff-Wald states that  $E[M] \leq (1 + \epsilon)500$  for  $\epsilon \approx 0.128$  (so  $(1 + \epsilon)500 \approx 564$ ).

If we were to modify the experiment so that the number of trials  $T$  was set at 1000, the Chernoff-Wald bound would give the same conclusion, but in that case the *Chernoff* bound would also imply (for the same  $\epsilon$ ) that  $\Pr[M \geq (1 + \epsilon)500] < 1$

**LEMMA 4.1. (Wald's inequality)** *Let  $T \in \mathbb{N}_+$  be a random variable with  $E[T] < \infty$  and let  $X_1, X_2, \dots$  be a sequence of random variables. Let  $\mu, c \in \mathbb{R}$ . If*

$$E[X_t | T \geq t] \leq \mu \quad (\text{for } t \geq 1)$$

*and  $X_t \leq c$  (for  $t = 1, \dots, T$ ), then*

$$E(X_1 + X_2 + \dots + X_T) \leq \mu E(T).$$

*The claim also holds if each " $\leq$ " is replaced by " $\geq$ ".*

The condition " $X_t \leq c$ " is necessary. Consider choosing each  $X_t$  randomly to be  $\pm 2^t$  and letting  $T = \min\{t : X_t > 0\}$ . Then  $E[X_t | T \geq t] = 0$ , so taking  $\mu = 0$ , all conditions for the theorem except " $X_t < c$ "

<sup>2</sup>The Chernoff-Wald bound also plays a central role in the randomized-rounding interpretation of the Garg and Koenemann's recent multicommodity-flow algorithm [5]. This and the general connection between randomized rounding and greedy/Lagrangian-relaxation algorithms are explored in depth in the journal version of [16], which as of October 1999 is still being written.

are met. But the conclusion  $E[X_1 + X_2 + \dots + X_T] \leq 0$  does not hold, because  $X_1 + X_2 + \dots + X_T = 1$ .

The proof is just an adaptation of the proof of Wald's equation [1, p. 370]. The reader can skip it on first reading.

*Proof.* W.l.o.g. assume  $\mu = 0$ , otherwise apply the change of variables  $X'_t = X_t - \mu$  before proceeding. Define  $Y_t \doteq X_1 + X_2 + \dots + X_t$ . If  $E(Y_T) = -\infty$  then the claim clearly holds. Otherwise,

$$\begin{aligned} E(Y_T) &= \sum_{t=1}^{\infty} \Pr(T = t) E(Y_t | T = t) \\ &= \sum_{t=1}^{\infty} \sum_{s=1}^t \Pr(T = t) E(X_s | T = t). \end{aligned}$$

The sum of the positive terms above is at most  $\sum_t \sum_{s < t} \Pr(T = t) c = \sum_t \Pr(T = t) ct = E[cT] < \infty$ . Thus, the double sum is absolutely convergent so

$$\begin{aligned} &= \sum_{s=1}^{\infty} \sum_{t=s}^{\infty} \Pr(T = t) E(X_s | T = t) \\ &= \sum_{s=1}^{\infty} \Pr(T \geq s) E(X_s | T \geq s) \end{aligned}$$

This establishes the claim because  $E(X_s | T \geq s) \leq 0$ . The claim with " $\leq$ "s replaced by " $\geq$ "s follows via the change of variables  $X'_t \doteq -X_t$ ,  $\mu' = -\mu$ , and  $c' \doteq -c$ .

□

In many applications of Wald's, the random variables  $X_1, X_2, \dots$  will be independent, and  $T$  will be a *stopping time* for  $\{X_t\}$  — a random variable in  $\mathbb{N}_+$  such that the event " $T = t$ " is independent of  $\{X_{t+1}, X_{t+2}, \dots\}$ . In this case the following companion lemma facilitates the application of Wald's inequality:

**LEMMA 4.2.** *Let  $X_1, X_2, \dots$  be a sequence of independent random variables and let  $T$  be a stopping time for the sequence. Then  $E[X_t | T \geq t] = E[X_t]$ .*

*Proof.* Because  $X_t$  is independent of  $X_1, X_2, \dots, X_{t-1}$ , and the event " $T \geq t$ " (i.e.  $T \notin \{1, 2, \dots, t-1\}$ ) is determined by the values of  $X_1, X_2, \dots, X_{t-1}$ , it follows that  $X_t$  is independent of the event " $T \geq t$ ". □

Here is a statement of a standard Chernoff bound. For a proof see e.g. [14, 16].

Let  $\text{ch}(\epsilon) \doteq (1 + \epsilon) \ln(1 + \epsilon) - \epsilon \geq \epsilon \ln(1 + \epsilon)/2$ .  
For  $\epsilon \leq 1$ ,  $\text{ch}(\epsilon) \geq \epsilon^2/3$  and  $\text{ch}(-\epsilon) \geq \epsilon^2/2$ .

**LEMMA 4.3. (Chernoff Bound [14])**

Let  $X_1, X_2, \dots, X_k$  be a sequence of independent random variables in  $[0, 1]$  with  $E(\sum_i X_i) \geq \mu > 0$ . Let  $\epsilon > 0$ . Then  $\Pr[\sum_i X_i \geq \mu(1 + \epsilon)] < \exp(-\text{ch}(\epsilon)\mu)$ . For  $\epsilon < 1$ ,  $\Pr[\sum_i X_i \leq \mu(1 - \epsilon)] < \exp(-\text{ch}(-\epsilon)\mu)$ .

**THEOREM 4.1. (Chernoff-Wald Bound)**

For each  $i = 1, 2, \dots, m$ , let  $X_{i1}, X_{i2}, \dots$  be a sequence of random variables such that  $0 \leq X_{it} \leq 1$ . Let  $T \in \mathbf{N}_+$  be a random variable with  $E[T] < \infty$ . Let  $M = \max_{i=1}^m X_{i1} + X_{i2} + \dots + X_{iT}$ . Suppose

$$E[X_{it} | T \geq t; \{X_{jt} : j < t, 1 \leq i \leq m\}] \leq \mu$$

for all  $i, t$  for some  $\mu \in \mathbb{R}$ . Let  $\epsilon \geq 0$  satisfy

$$e^{-\text{ch}(\epsilon)} \max\{\mu E[T], E[M]/(1 + \epsilon)\} \leq 1/m.$$

Then

$$E[M] \leq (1 + \epsilon)\mu E[T].$$

The claim also holds with the following replacements: “ $\min_i$ ” for “ $\max_i$ ”; “ $\geq \mu$ ” for “ $\leq \mu$ ”; “ $E[M] \geq$ ” for “ $E[M] \leq$ ”; and, for each “ $\epsilon$ ”, “ $-\epsilon$ ” (except in “ $\epsilon \geq 0$ ”).

In the case when  $T$  is constant, the Chernoff bound implies  $\Pr[M \geq (1 + \epsilon)\mu T] < 1$  for  $\epsilon$  as above.

The first-time reader can skip the following proof.

*Proof.* For  $t = 0, \dots, T$ , let  $Y_{it} \doteq X_{i1} + X_{i2} + \dots + X_{it}$  ( $i = 1, \dots, m$ ) and

$$Z_t \doteq \log_{1+\epsilon} \sum_{i=1}^m (1 + \epsilon)^{Y_{it}}.$$

Note that for each  $i$ ,  $Z_T \geq \log_{1+\epsilon} (1 + \epsilon)^{Y_{iT}} = Y_{iT}$ . Thus  $M \leq Z_T$ . We use Wald’s inequality to bound  $E[Z_T]$ . Fix any  $t > 0$ . Let  $y_{it} = (1 + \epsilon)^{Y_{i,t-1}}$ . Then

$$\begin{aligned} Z_t - Z_{t-1} &= \log_{1+\epsilon} \frac{\sum_i y_{it}(1 + \epsilon)^{X_{it}}}{\sum_i y_{it}} \\ &\leq \log_{1+\epsilon} \frac{\sum_i y_{it}(1 + \epsilon X_{it})}{\sum_i y_{it}} \\ &= \log_{1+\epsilon} \left[ 1 + \epsilon \frac{\sum_i y_{it} X_{it}}{\sum_i y_{it}} \right] \\ &< \frac{\epsilon}{\ln(1 + \epsilon)} \frac{\sum_i y_{it} X_{it}}{\sum_i y_{it}}. \end{aligned}$$

The first inequality follows from  $(1 + \epsilon)^z \leq 1 + \epsilon z$  for  $0 \leq z \leq 1$ . The last inequality follows from  $\log_{1+\epsilon} 1 + z = \ln(1 + z)/\ln(1 + \epsilon) < z/\ln(1 + \epsilon)$  for

$z \neq 0$ . Thus, conditioned on the event  $T \geq t$  and the values of  $Y_{i,t-1}$  ( $1 \leq i \leq m$ ),

$$E[Z_t - Z_{t-1}] < \frac{\epsilon}{\ln(1 + \epsilon)} \frac{\sum_i y_{it} \mu}{\sum_i y_{it}}.$$

This implies that  $E[Z_t - Z_{t-1} | T \geq t] < \mu\epsilon/\ln(1 + \epsilon)$ . Using  $Z_0 = \log_{1+\epsilon} m$  and Wald’s inequality,

$$E[M] \leq E[Z_T] < \log_{1+\epsilon} m + E[T]\mu\epsilon/\ln(1 + \epsilon).$$

By algebra, the above together with the assumption on  $\epsilon$  imply that  $E[M] \leq (1 + \epsilon)\mu E[T]$ .

The proof of the claim for the minimum is essentially the same, with each “ $\epsilon$ ” replaced by “ $-\epsilon$ ” and reversals of appropriate inequalities. In verifying this, note that  $\log_{1-\epsilon}$  is a decreasing function.  $\square$

In many applications of Chernoff-Wald, each  $X_{it}$  will be independent of  $\{X_{j\ell} : \ell < t, 1 \leq j \leq m\}$ , and  $T$  will be a stopping time for  $\{X_{it}\}$  — a random variable in  $\mathbf{N}_+$  such for any  $t$  the event “ $T = t$ ” is independent of  $\{X_{i\ell} : \ell > t, 1 \leq i \leq m\}$ . Then by an argument similar to the proof of Lemma 4.2, we have:

**LEMMA 4.4.** For  $1 \leq i \leq m$ , let  $X_{i1}, X_{i2}, \dots$  be a sequence of random variables such that each  $X_{it}$  is independent of  $\{X_{j\ell} : \ell < t, 1 \leq j \leq m\}$ . Let  $T$  be a stopping time for  $\{X_{it}\}$ . Then  $E[X_{it} | T \geq t; \{X_{jt} : j < t, 1 \leq i \leq m\}] = E[X_{it}]$ .

## 5 Randomized rounding for facility location.

We use Wald’s inequality to analyze a natural randomized rounding scheme for facility location.

**GUARANTEE 5.1.** Let  $F$  be the output of the facility-location rounding scheme in Fig. 2 given input  $x$ . Let  $\Delta_{fx}$  be the number of customers  $c$  such that  $x(f, c) > 0$ . Then  $E_F[\text{dist}(F) + \text{cost}(F)]$  is at most  $\text{dist}(x) + \sum_f \text{cost}(f)x(f)H_{\Delta_{fx}}$ .

*Proof.* Observe the following basic facts about each iteration of the outer loop:

1. The probability that a given customer  $c$  is assigned to a particular facility  $f$  in this iteration is  $x(f, c)/|x|$ .
2. The probability that  $c$  is assigned to *some* facility is  $\sum_f x(f, c)/|x| \geq 1/|x|$ .
3. Given that  $c$  is assigned, the probability of it being assigned to a particular  $f$  is  $x(f, c)$ .

**input:** fractional facility location solution  $x$ .  
**output:** random solution  $F \subset \mathcal{F}$  s.t.  $E_F[\text{dist}(F) + \text{cost}(F)] \leq \text{dist}(x) + H_\Delta \text{cost}(x)$ .

1. Repeat until all customers are assigned:
2. Choose a single facility  $f$  at random so that  $\Pr(f \text{ chosen}) = x(f)/|x|$ .
3. For each customer  $c$  independently with probability  $x(s, e)/x(s)$ :
4. Assign (or, if  $c$  was previously assigned, reassign)  $c$  to  $f$ .
5. Return the set containing those facilities having customers assigned to them.

Figure 2: Facility-location rounding scheme. Note  $|x| \doteq \sum_s x(s)$ .

To bound  $E[\text{dist}(F)]$ , it suffices to bound the expected cost of the assignment chosen by the algorithm. For a given pair  $(f, c)$ , what is the probability that  $f$  is assigned to  $c$ ? By the third fact above, this is  $x(f, c)$ . Thus,  $E[\text{dist}(F)] \leq \sum_{f, c} \Pr(c \text{ assigned to } f) \text{dist}(f, c) = \text{dist}(x)$ .

To finish, we show  $\Pr(f \in F) \leq H_{\Delta_{f_x}} x(f)$ . This suffices because it implies that  $E[\text{cost}(F)]$  equals  $\sum_f \Pr(f \in F) \text{cost}(f) \leq \sum_f H_{\Delta_{f_x}} x(f) \text{cost}(f)$ .

Fix a facility  $f$ . Call the customers  $c$  such that  $x(f, c) > 0$  the “fractional customers of  $f$ ”. Let random variable  $T$  be the number of iterations before all these customers are assigned.

**claim 1:**  $\Pr(f \in F) \leq E[T]x(f)/|x|$ .

**proof:** Define  $X_t$  to be the indicator variable for the event “ $f$  is first chosen in iteration  $t$ ”.

As  $E[X_t | T \geq t] \leq x(f)/|x|$ , by Wald’s inequality  $\Pr(f \in F) = E[X_1 + X_2 + \dots + X_T] \leq E[T]x(f)/|x|$ . This proves the claim.

Recall that  $\Delta_{f_x}$  is the number of fractional customers of  $f$ . To finish the proof, it suffices to show:

**claim 2:**  $E[T] \leq |x|H_{\Delta_{f_x}}$

**proof:** Define  $u_t$  to be the number of fractional customers of  $f$  not yet assigned after iteration  $t$  ( $0 \leq t \leq T$ ). (Recall  $H_0 \doteq 0$  and  $H_i \doteq 1 + 1/2 + \dots + 1/i$ .) Then provided  $t \leq T$ ,  $H_{u_t} - H_{u_{t+1}}$  is

$$\frac{1}{u_t} + \frac{1}{u_t - 1} + \dots + \frac{1}{u_{t+1} - 1} \geq \frac{u_t - u_{t+1}}{u_t}.$$

The expectation of the right-hand side is at least  $1/|x|$  because each customer is assigned in each iteration with probability at least  $1/|x|$ . Since  $H_{u_t}$  is  $H_{\Delta_{f_x}}$  when  $t = 0$  and decreases by at least  $1/|x|$  in expectation each iteration, by Wald’s inequality, it follows that  $E[H_{\Delta_{f_x}} - H_{u_T}] \geq E[T]/|x|$ . Since  $H_{u_T} = 0$ , the claim follows.  $\square$

The randomized scheme can easily be derandomized.

**COROLLARY 5.1.** *There is a polynomial-time  $[d + H_\Delta k]$ -approximation algorithm for uncap. facility location.*

## 6 Greedy algorithm for weighted $k$ -medians.

The  $k$ -medians rounding scheme takes a fractional  $k$ -medians solution  $(x, k, d)$  and an  $\epsilon > 0$  and outputs a random solution  $F$ . The scheme is the same as the facility-location rounding scheme in Fig. 2 except for the termination condition. The modified algorithm terminates after the first iteration in which the facility cost exceeds  $k[\ln(n + n/\epsilon)]$  or all customers are assigned with assignment cost less than  $d(1 + \epsilon)$ . We analyze this rounding scheme using Wald’s inequality and then derandomize it to obtain a greedy algorithm (Fig. 3).

**GUARANTEE 6.1.** *Let  $F$  be the output of the weighted  $k$ -medians rounding scheme given input  $x$ . Then  $\text{cost}(F) \leq k \ln(n + n/\epsilon) + \max_f \text{cost}(f)$  and with positive probability  $\text{dist}(F) < d(1 + \epsilon)$ .*

**Proof.** The bound on  $\text{cost}(F)$  always holds due to the termination condition of the algorithm.

Let random variable  $T$  be the number of iterations of the rounding scheme. Let random variable  $u_t$  be the number of not-yet-assigned customers at the end of round  $t$  ( $0 \leq t \leq T$ ). By fact 2 in the proof of Guarantee 5.1,  $E[u_t | u_{t-1} \wedge T \geq t] = (1 - 1/|x|)u_{t-1}$ .

Define random variable  $d_t$  to be the total cost of the current (partial) assignment of customers to facilities at the end of round  $t$ . Because each customer is reassigned with probability  $1/|x|$  in each round, it is not hard to show that  $E[d_t | d_{t-1} \wedge T \geq t] = (1 - 1/|x|)d_{t-1} + d/|x|$ .

Define random variable  $c_t$  to be the total cost of the facilities chosen so far at the end of round  $t$ . In each iteration,  $E[\text{cost}(f)] = k/|x|$ , so that  $E[c_t | c_{t-1} \wedge T \geq t] = c_{t-1} + k/|x|$ .

Define  $\phi_t \doteq c_t/k + \ln[u_t + (d_t/d - 1)/(1 + \epsilon)]$ . Then using  $\ln z \leq z - 1$  and the two facts established in the preceding three paragraphs, a calculation shows  $E[\phi_t - \phi_{t-1} | d_{t-1}, u_{t-1}, c_{t-1}] \leq 0$ . By Wald’s inequality, this implies  $E[\phi_T] \leq \phi_0 < \ln n$ . Thus, with positive probability,  $\phi_T \leq \ln n$ . Assuming this event occurs, we will show that at the end all elements are covered and the assignment cost is not too high.

**input:**  $\text{dist}()$ ,  $\text{cost}()$ ,  $\epsilon$ ,  $d$ ,  $n$ .  
**output:** Set  $F$  of facilities s.t.  $\text{cost}(F) \leq k \ln(n + n/\epsilon) + \max_f \text{cost}(f)$  and  $\text{dist}(F) \leq (1 + \epsilon)d$ .

1. Define  $\phi(c, f) \doteq \text{dist}(c, f)/d(1 + \epsilon)$ . ...  $c$ 's contrib. to  $\bar{u}_t + \bar{d}_t/d(1 + \epsilon)$  if  $c$  assigned to  $f$ .
2. For each customer  $c$  do:  $f_c \leftarrow \text{none}$ . ...  $f_c$  is the facility  $c$  is currently assigned to
3. Define  $\phi(c, \text{none}) \doteq 1$ . ... contribution is 1 if  $c$  is unassigned
4. Repeat until all customers are assigned with assignment cost  $\leq d(1 + \epsilon)$ :
5. For each facility  $f$  define  $C_f \doteq \{c : \phi(c, f_c) > \phi(c, f)\}$ .
6. Choose a facility  $f$  to maximize  $[\sum_{c \in C_f} \phi(c, f_c) - \phi(c, f)] / \text{cost}(f)$ .
7. For each  $c \in C_f$  do:  $f_c \leftarrow f$ . ... Assign  $c$  to  $f$ .
8. Return the set of chosen facilities.

Figure 3: The greedy  $k$ -medians algorithm.

If the rounding scheme terminates because all elements are covered and the assignment cost is less than  $(1 + \epsilon)d$ , then clearly the performance guarantee holds. Otherwise the algorithm terminates because the facility cost  $c_T$  exceeds  $\ln(n + n/\epsilon)k$ . This lower bound on the size and the occurrence of the event “ $\phi_T \leq \ln n$ ” imply that  $u_T < 1$  and  $d_T < (1 + \epsilon)d$ .  $\square$

Next we apply the method of conditional probabilities. Let  $T$ ,  $d_t$ ,  $u_t$ ,  $c_t$ , and  $\phi_t$  ( $0 \leq t \leq T$ ) be defined as in the proof of Guarantee 6.1 for the  $k$ -medians rounding scheme. That proof showed that  $E[\phi_T] < \ln n$ , and that if  $\phi_T < \ln n$  then  $F$  meets the performance guarantee.

To obtain the greedy algorithm, in each iteration we replace the random choices by deterministic choices. Let  $\bar{d}_t$ ,  $\bar{u}_t$ , and  $\bar{c}_t$  denote, respectively, the assignment cost, number of unassigned elements, and facility cost at the end of the  $t$ th iteration of the greedy algorithm (analogous to  $d_t$ ,  $u_t$ , and  $c_t$  for the randomized algorithm). The greedy algorithm will make its choices in a way that maintains the invariant

$$E[\phi_T | d_t = \bar{d}_t \wedge u_t = \bar{u}_t \wedge c_t = \bar{c}_t] < \ln n.$$

Note that the expectation above is with respect to the random experiment. That is, the invariant says that if, starting from the current configuration, the remaining choices were to be made randomly, then (in expectation)  $\phi$  would end up less than  $\ln n$ .

Define  $\bar{\phi}_t \doteq \bar{c}_t/k + \ln[\bar{u}_t + (\bar{d}_t/d - 1)/(1 + \epsilon)]$  (analogous to  $\phi_t$  for the randomized algorithm). The proof of Guarantee 6.1 easily generalizes to show  $E[\phi_T | d_t = \bar{d}_t \wedge u_t = \bar{u}_t \wedge c_t = \bar{c}_t] \leq \bar{\phi}_t$ . Thus, it suffices to maintain the invariant  $\bar{\phi}_t < \ln n$ . Since  $\bar{\phi}_0 = \phi_0 < \ln n$ , the invariant holds initially.

During each iteration  $t$ , the algorithm chooses a facility  $f$  and assigns it a set of customers  $C$  so that  $\bar{\phi}_t \leq \bar{\phi}_{t-1}$ . A calculation shows  $\bar{\phi}_t - \bar{\phi}_{t-1}$  is less than

$$\frac{\text{cost}(f)}{k} - \frac{\bar{u}_{t-1} - \bar{u}_t + (\bar{d}_{t-1}/d - \bar{d}_t/d)/(1 + \epsilon)}{\bar{u}_{t-1} + (\bar{d}_{t-1}/d - 1)/(1 + \epsilon)}.$$

It suffices to choose  $f$  and  $C$  so that the above is non-positive. Whatever  $\bar{u}_{t-1}$ ,  $\bar{c}_{t-1}$ , and  $\bar{d}_{t-1}$  are, if  $f$  and  $C$  are chosen randomly, then the expectation of the above is zero. Thus, there is some choice of  $f$  and  $C$  which makes it non-positive. Thus it suffices to choose  $f$  and  $F$  to maximize

$$\frac{\bar{u}_{t-1} - \bar{u}_t + (\bar{d}_{t-1}/d - \bar{d}_t/d)/(1 + \epsilon)}{\text{cost}(f)}.$$

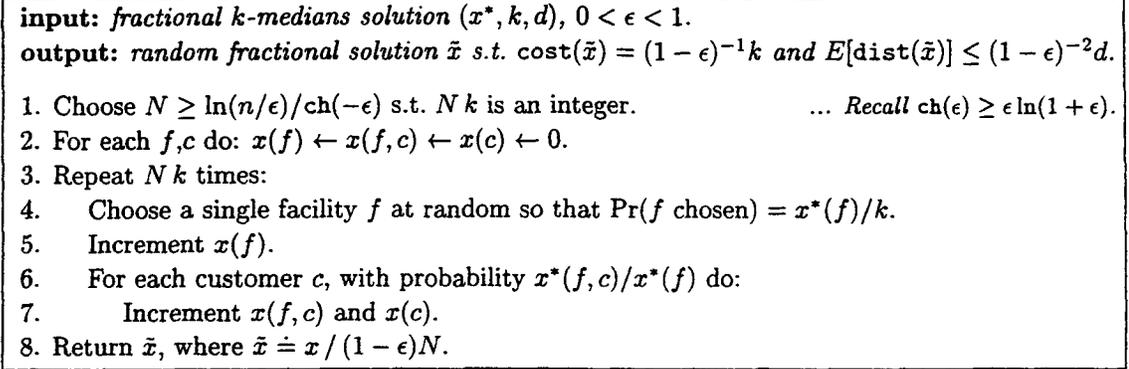
The algorithm considers each facility  $f$ ; for each  $f$ , it determines the best set  $C$  of customers to assign. The algorithm is shown in Fig. 3.

The termination condition in the algorithm differs from the one in the rounding scheme, but the modified termination condition suffices because it follows from the analysis that whatever  $k$  is, the algorithm will terminate no later than the first iteration such that the facility cost exceeds  $k \ln(n + n/\epsilon)$ . By the derivation,

**GUARANTEE 6.2.** *Given  $\epsilon$ , and  $d$  such that a fractional solution of cost  $k$  and assignment cost at most  $d$  exists, the greedy weighted  $k$ -medians algorithm (Fig. 3) returns a solution  $F$  such that  $\text{dist}(F) \leq (1 + \epsilon)d$  and  $\text{cost}(F) \leq k \ln(n + n/\epsilon) + \max_f \text{cost}(f)$ .*

Without loss of generality (since we are approximately solving the IP)  $\max_f \text{cost}(f) \leq k$ . For the unweighted problem, the number of iterations is  $O(k \ln(n/\epsilon))$ . No facility is chosen twice, so the number of iterations is always at most  $m$ , the number of facilities.

**COROLLARY 6.1.** *Let  $0 < \epsilon < 1$ . The weighted  $k$ -medians problem has a  $[(1 + \epsilon)d; (1 + \ln(n + n/\epsilon)k)]$ -approximation algorithm that runs in  $O(m)$  linear-time iterations, or  $O(k \ln(n/\epsilon))$  iterations for the unweighted problem.*

Figure 4: Fractional  $k$ -medians rounding scheme.

### 7 Lagrangian relaxation for $k$ -medians.

In this section we derive and analyze a Lagrangian-relaxation,  $[(1 + \epsilon)^2 d; (1 + \epsilon)k]$ -approximation algorithm for the fractional unweighted  $k$ -medians problem. The rounding scheme is shown in Fig. 4. We use the Chernoff and Markov bounds to bound the probability of failure.

**GUARANTEE 7.1.** *Let  $\tilde{x}$  be the output of the fractional  $k$ -medians rounding scheme. Then with positive probability  $\tilde{x}$  has  $\text{cost}(\tilde{x}) \leq (1 - \epsilon)^{-1}k$  and  $\text{dist}(\tilde{x}) \leq (1 - \epsilon)^{-2}d$ .*

*Proof.* Recall  $k = \text{cost}(x^*) = |x^*|$  and  $d = \text{dist}(x^*)$ . The bound on the cost always holds, because each of the  $kN$  iterations adds 1 to  $\text{cost}(x)$ . It remains to show that with positive probability, after the final iteration,  $\text{dist}(x) \leq (1 - \epsilon)^{-1}Nd$  and each  $x(c) \geq (1 - \epsilon)N$ .

Since each iteration increases  $\text{dist}(x)$  by  $d/k$  in expectation, finally  $E[\text{dist}(x)] \leq (Nk)d/k = dN$ . By the Markov bound,  $\Pr[\text{dist}(x) \geq dN/(1 - \epsilon)] \leq 1 - \epsilon$ .

For any customer  $c$ ,  $x(c)$  is the sum of  $kN$  independent 0-1 random variables each with expectation at least  $1/k$ , so by the Chernoff bound,  $\Pr[x(c) \leq (1 - \epsilon)N] < \exp(-\text{ch}(-\epsilon)N)$ , which is at most  $\epsilon/n$  by the choice of  $N$ .

By the naive union bound,  $\Pr[\text{dist}(x) \geq dN/(1 - \epsilon) \vee \min_c x(c) \leq (1 - \epsilon)N] < (1 - \epsilon) + n(\epsilon/n) = 1$ .  $\square$

Next we sketch how the method of conditional expectations yields the algorithm shown in Fig. 5. The proof of Guarantee 7.1 implicitly bounds the probability of failure by the expectation of

$$\frac{\text{dist}(x)}{dN/(1 - \epsilon)} + \sum_c \frac{(1 - \epsilon)^{x(c)}}{(1 - \epsilon)^{(1 - \epsilon)N}}$$

and it shows that the expectation is less than 1. An upper bound (called a ‘‘pessimistic estimator’’ [14]) of

the conditional expectation of the final value of the above, given the current value of  $x$  and the number  $t$  of remaining iterations, is

$$\widehat{\Phi}(x, t) \doteq \frac{\text{dist}(x) + td/k}{dN/(1 - \epsilon)} + \sum_c \frac{(1 - \epsilon)^{x(c)} e^{-t\epsilon/k}}{(1 - \epsilon)^{(1 - \epsilon)N}}$$

The algorithm chooses  $f$  and  $C$  in each iteration in order to minimize the increase in the above quantity, which consequently stays less than 1.

**GUARANTEE 7.2.** *The fractional  $k$ -medians algorithm in Fig. 5 returns a fractional solution  $\hat{x}$  having  $\text{cost}(\hat{x}) \leq (1 - \epsilon)^{-1}k$  and  $\text{dist}(\hat{x}) \leq (1 - \epsilon)^{-2}d$ .*

*Proof.* (Sketch.) Let  $\widehat{\Phi}$  be as defined above. The algorithm maintains the invariant that with  $t$  iterations remaining,  $\widehat{\Phi}(x, t) < 1$ . It is straightforward to verify that the invariant is initially true, and that if it is true at the end, then the performance guarantee is met. We verify that the invariant is maintained at each step. The increase in  $\widehat{\Phi}$  in a single iteration is proportional to

$$\sum_{c \in C} \text{dist}(f, c) - \frac{d}{k} + \sum_c y(c) e^{\epsilon/k} (1 - \epsilon)^{c \in C} - \sum_c y(c)$$

where  $y(c) = \alpha(1 - \epsilon)^{x(c)}$  before the iteration for suitably chosen scalar  $\alpha \geq 0$ . If  $f$  and  $C$  were chosen randomly as in the rounding scheme, the expectation of the above would be at most 0. The choice made by the algorithm minimizes the above quantity, therefore the algorithm maintains the invariant.  $\square$

**COROLLARY 7.1.** *Let  $0 < \epsilon < 1$ . The fractional  $k$ -medians problem has a  $[(1 + \epsilon)d, (1 + \epsilon)k]$ -approximation algorithm that runs in  $O(k \ln(n/\epsilon)/\epsilon^2)$  linear-time iterations.*

**input:** fractional  $k$ -medians instance,  $0 < \epsilon < 1$ ,  $k$ ,  $d$ .  
**output:** fractional solution  $\hat{x}$  s.t.  $\text{cost}(\hat{x}) \leq (1 - \epsilon)^{-1}k$  and  $\text{dist}(\hat{x}) \leq (1 - \epsilon)^{-2}d$ .

1. Choose  $N \geq \ln(n/\epsilon)/\text{ch}(-\epsilon)$  such that  $Nk$  is an integer. ... Recall  $\text{ch}(-\epsilon) \geq \epsilon^2/2$ .
2. For each  $f, c$  do:  $x(f) \leftarrow x(f, c) \leftarrow x(c) \leftarrow 0$ ;  $y(c) = \epsilon(1 - \epsilon)^{-1}Nde^{\text{ch}(-\epsilon)N}$ .
3. Repeat  $Nk$  times:
  4. For each  $c$  do: set  $y(c) \leftarrow y(c)e^{\epsilon/k}$ .
  5. Choose a single facility  $f$  and a set of customers  $C$  to maximize  $\sum_{c \in C} y(c) - \text{dist}(f, c)$ .
  6. Increment  $x(f)$ .
  7. For each  $c \in C$  do: Increment  $x(f, c)$  and  $x(c)$ , and set  $y(c) \leftarrow (1 - \epsilon)y(c)$ .
8. Return  $\hat{x}$ , where  $\hat{x} = x / (1 - \epsilon)N$ .

Figure 5: Lagrangian-relaxation algorithm for fractional  $k$ -medians. In step 4, it suffices to choose the best set of the form  $C = \{c : y(c) > \text{dist}(f, c)\}$  for some  $f$ . This can be done in linear time.

### 8 Lagrangian relaxation for facility location.

The rounding scheme for fractional facility location is the same as the rounding scheme for fractional  $k$ -medians in Fig. 4, except for the termination condition. The modified algorithm terminates after the first iteration where each  $x(c) \geq (1 - \epsilon)N$ , where  $N$  is chosen to be at least  $\ln(n)/\text{ch}(-\epsilon)$  s.t.  $(1 - \epsilon)N$  is an integer. We use the Chernoff-Wald bound to analyze the scheme.

**GUARANTEE 8.1.** *Let  $\tilde{x}$  be the output of the fractional facility-location rounding scheme as described above. Then  $\tilde{x}$  is a fractional solution to the facility location LP and  $E[\text{cost}(\tilde{x}) + \text{dist}(\tilde{x})] \leq (1 - \epsilon)^{-1}(d + k)$ .*

*Proof.* The termination condition ensures that all customers are adequately covered. It remains to bound  $E[\text{cost}(\tilde{x}) + \text{dist}(\tilde{x})]$ .

Let r.v.  $T$  denote the number of iterations of the rounding scheme. In each iteration, the expected increase in  $\text{cost}(x) + \text{dist}(x)$  is  $k/|x^*| + d/|x^*|$ . By Wald's inequality, at termination,  $E[\text{cost}(x) + \text{dist}(x)] \leq E[T](k + d)/|x^*|$ . It remains to show  $E[T] \leq N|x^*|$ . (Recall that  $\tilde{x} = x/(1 - \epsilon)N$ .)

For any customer  $c$ , the probability that  $x(c)$  is incremented in a given iteration is at least  $1/|x^*|$ , independently of the previous iterations. Let r.v.  $M \doteq \min_c x(c)$  at the end. Note that, by the choice of  $N$ , in fact  $M = (1 - \epsilon)N$ .

By the Chernoff-Wald bound,  $(1 - \epsilon)N = E[M] \geq (1 - \epsilon)E[T]/|x^*|$  (provided  $\text{ch}(-\epsilon) \geq \ln(m)(1 - \epsilon)/E[M]$ , which indeed holds by the choice of  $N$ ). Rewriting gives  $E[T] \leq N|x^*|$ .  $\square$

Next we sketch how applying the method of conditional expectations gives the Lagrangian-relaxation algorithm shown in Fig. 6. Below we let  $x_f$  denote

the value of  $x$  after the final iteration of the algorithm and  $x$  denote the value at the ‘‘current’’ iteration. The analysis of the rounding scheme shows that  $E[\text{cost}(x_f) + \text{dist}(x_f)] \leq N(k + d)$ . The conditional expectation of  $\text{cost}(x_f) + \text{dist}(x_f)$  at the end, given the current  $x$ , is  $\text{cost}(x) + \text{dist}(x) + E[t|x](k + d)/|x^*|$ , where random variable  $t$  is the number of iterations left.

The proof of Chernoff-Wald, in this context, argues that the quantity  $\tilde{M}(x) \doteq \log_{1-\epsilon} \sum_c (1 - \epsilon)^{x(c)}$  is  $\log_{1-\epsilon} n$  initially, at most  $(1 - \epsilon)N$  finally, and decreases in expectation at least  $-\epsilon/(|x^*| \ln(1 - \epsilon))$  in each iteration. An easy generalization of the argument shows  $E[t|x]$  is at most  $\frac{(1 - \epsilon)N - \tilde{M}(x)}{-\epsilon/(|x^*| \ln(1 - \epsilon))}$ . This gives us our pessimistic estimator:  $E[\text{cost}(x_f) + \text{dist}(x_f) | x] \leq \hat{\Phi}(x)$  where

$$\hat{\Phi}(x) \doteq \text{cost}(x) + \text{dist}(x) + (k + d) \frac{(1 - \epsilon)N - \tilde{M}(x)}{-\epsilon/\ln(1 - \epsilon)}.$$

The algorithm chooses  $f$  and  $C$  to keep  $\hat{\Phi}$  from increasing (although not necessarily to *minimize*  $\hat{\Phi}$ ) at each round.

**GUARANTEE 8.2.** *Let  $\hat{x}$  be the output of the algorithm shown in Fig. 6. Then  $\hat{x}$  is a fractional solution to the facility location LP and  $\text{cost}(\hat{x}) + \text{dist}(\hat{x}) < (1 - \epsilon)^{-1} \min_x \text{dist}(x) + \text{cost}(x)$ , where  $x$  ranges over all fractional solutions.*

*Proof.* (Sketch) Define  $\hat{\Phi}$  as above. The algorithm maintains the invariant  $\hat{\Phi}(x) \leq (k + d)N$ . A calculation<sup>3</sup> shows that the invariant is initially true by the choice of  $N$ . Clearly if the invariant is true at the end the

<sup>3</sup>This proof is an adaptation of part of the Chernoff-Wald proof to this context. For further details on parts marked with this footnote, see that proof.

**input:** fractional facility-location instance,  $0 < \epsilon < 1$ .  
**output:** fractional solution  $\hat{x}$  s.t.  $\text{cost}(\hat{x}) + \text{dist}(\hat{x}) \leq (1 - \epsilon)^{-1}(d + k)$ .

1. Choose  $N \geq \ln(n)/\text{ch}(-\epsilon)$  such that  $N(1 - \epsilon)$  is an integer. ... Recall  $\text{ch}(-\epsilon) \geq \epsilon^2/2$ .
2. For each  $f, c$  do:  $x(f) \leftarrow x(f, c) \leftarrow x(c) \leftarrow 0$ ;  $y(c) \leftarrow 1$ .
3. Repeat until each  $x(c) \geq (1 - \epsilon)N$ :
4. Choose a single facility  $f$  and a set of customers  $C$  to maximize  $\frac{\sum_{c \in C} y(c)}{\text{cost}(f) + \sum_{c \in C} \text{dist}(f, c)}$ .
5. Increment  $x(f)$ .
6. For each  $c \in C$  do:
7. Increment  $x(f, c)$  and  $x(c)$ , and set  $y(c) \leftarrow (1 - \epsilon)y(c)$ . If  $x(c) > N$  set  $y(c) \leftarrow 0$ .
8. Return  $\hat{x}$ , where  $\hat{x} \doteq x / (1 - \epsilon)N$ .

Figure 6: Lagrangian-relaxation algorithm for fractional facility location. In step 4, it suffices to choose the best set of the form  $C = \{c : y(c)/\text{dist}(f, c) \geq \lambda\}$  for some  $\lambda$ .

performance guarantee holds. In a given iteration, the increase in  $\hat{\Phi}$  is at most<sup>3</sup>  $(k + d)\epsilon$  times

$$\frac{\text{cost}(f) + \sum_{c \in C} \text{dist}(f, c)}{k + d} - \frac{\sum_{c \in C} y(c)}{\sum_c y(c)}$$

where  $y(c) = (1 - \epsilon)^{x(c)}$ . If  $f$  and  $C$  were chosen randomly as in the rounding scheme, the expectation of the above quantity would be non-positive.<sup>3</sup> Thus, to keep it non-positive, it suffices to choose  $f$  and  $c$  to maximize

$$\frac{\sum_{c \in C} y(c)}{\text{cost}(f) + \sum_{c \in C} \text{dist}(f, c)}$$

which is what the algorithm does. □

In each iteration, at least one customer  $c$  with  $x(c) \leq N$  has  $x(c)$  incremented. Thus the number of iterations is  $O(n \ln(n)/\epsilon^2)$ . Each iteration can be implemented in linear times  $O(\ln(n))$  time. Thus,

**COROLLARY 8.1.** *Let  $0 < \epsilon < 1$ . Fractional facility location has a  $[(1 + \epsilon)(d + k)]$ -approximation algorithm that runs in  $O(n \ln(n)/\epsilon^2)$  iterations, each requiring time linear in the input size times  $\ln n$ .*

## 9 Further directions

Is there a greedy  $[d + H_\Delta k]$ -approximation algorithm for facility location? A  $[(1 + \epsilon)d; (1 + \ln(n + n/\epsilon))k]$ -approximation algorithm for weighted  $k$ -medians? A  $[d, (1 + \epsilon)k]$ - or  $[(1 + \epsilon)d, k]$ -approximation algorithm for fractional  $k$ -medians? A Lagrangian-relaxation algorithm for fractional weighted  $k$ -medians?

The running times of all of the algorithms here can probably be improved using techniques similar to the one that Fleischer applied to improve Garg and

Konemann's multicommodity flow algorithm [4], or (depending on the application) using standard data structures.

In practice, changing the objective function of the LP relaxation of the IP to better reflect the performance guarantee might be worthwhile. For example, if one is going to randomly round a fractional solution  $x$  to the facility-location LP, it might be better to minimize  $\text{dist}(x) + \sum_f \text{cost}(f)x(f)H_\Delta$ , rather than  $\text{dist}(x) + \text{cost}(x)$ . This gives a performance guarantee that is provably as good, and may allow the LP to compensate for the fact that the difficulty of approximating the various components of the cost varies.

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### A $K$ -medians via $k$ -set cover via PST.

For completeness, we discuss a relation between fractional  $k$ -medians and the mixed/packing covering framework of Plotkin, Shmoys, and Tardos (PST) [13]. First we consider the  $k$ -set cover problem — a variant of weighted set cover in which each set  $s$  is given a “distance”  $\text{dist}(s) \in \mathbb{R}_+$ , and the goal is to choose a cover (a collection of sets containing all elements) of size at most  $k$ , minimizing the total distance.

We formulate the *decision* problem (given  $d$ , is there a set cover of size at most  $k$  and distance at most  $d$ ?) as a mixed packing/covering problem. Let  $P \doteq \{x : \sum_s x_s \leq K\}$ . For  $x \in P$  define  $x_e \doteq \sum_{s \ni e} x_s$  and  $\text{dist}(x) \doteq \sum_s x_s \text{dist}(s)$ . Then the fractional  $K$ -set cover problem is the packing/covering problem  $\exists x \in P : (\forall e) x_e \geq 1; \text{dist}(x) \leq d$ .

We can solve this using the PST algorithm as follows.

The input to that algorithm is  $(\text{dist}(), k, d, \epsilon)$ . We can use it to compute an approximate solution  $x$  such that  $(\forall e) x_e \geq 1 - \epsilon$  and  $\text{dist}(x) \leq (1 + \epsilon)d$  (provided the original problem is feasible). We scale  $x$ , multiplying it by  $1 + O(\epsilon)$ , to get the final output.

With care, we can show that to implement the PST algorithm, it suffices to have a subroutine that, given a vector  $\alpha$ , returns  $x \in P$  minimizing  $\text{dist}(x) - \sum_e \alpha_e x_e$ . An optimal  $x$  can be found by enumerating the sets and choosing the set  $s$  that minimizes  $\text{dist}(s) - \sum_{e \in s} \alpha_e$ .

The running time of the PST algorithm is dominated by the time spent in this subroutine. The subroutine is called  $O(\rho \ln(m)/\epsilon^2)$  times, where  $m$  is the number of elements and  $\rho$  is the *width* of the problem instance, which in this case is  $k \max_s \{\text{dist}(s)/d, 1\}$ . Thus,

**COROLLARY A.1.** *The fractional  $k$ -set cover decision problem reduces to a mixed packing/covering problem of width  $\rho = k \max_s \{\text{dist}(s)/d, 1\}$ . If a problem instance is feasible, the algorithm of [13] yields a fractional solution  $x$  with  $|x| \leq (1 + \epsilon)k$  and  $\text{dist}(x) \leq (1 + \epsilon)d$  in time linear in the input size times  $O(\rho \ln(m)/\epsilon^2)$ .*

In many cases, we can assume without loss of generality that  $\max_s \text{dist}(s) \leq d$ , in which case the width is  $k$ . Except for the fact that this is a decision procedure, this is comparable to Corollary 7.1. (Although that bound requires no assumption about  $d$ .)

Next we sketch how weighted  $k$ -medians reduces to  $k$ -set cover. We adapt Hochbaum’s facility-location-to-set-cover reduction. Fix a weighted  $k$ -medians instance with  $n$  facilities and  $m$  customers  $\mathcal{C}$ . Construct an (exponentially large) family of sets as follows. For each facility  $f$  and subset  $C$  of customers, define a set  $S_{fC} = C$ , with  $\text{dist}(S_{fC}) \doteq \sum_{c \in C} \text{dist}(f, c)$  and  $\text{cost}(S_{fC}) \doteq \text{cost}(f)$ . Then each  $k$ -medians solution corresponds to a  $k$ -set cover, and vice versa. The bijection preserves  $\text{dist}$ , and extends naturally to the fractional problems as well.

Even though the resulting fractional  $k$ -set is exponentially large, we can still solve it efficiently using PST provided we have a subroutine that, given a vector  $\alpha$ , efficiently finds a facility  $f$  and set of customers  $C$  minimizing  $\sum_{c \in C} \text{dist}(f, c) - \alpha_c$ . This  $C$  and  $f$  can in fact be found by choosing the facility  $f$  minimizing  $\sum_{c \in C_f} \text{dist}(f, c) - \alpha_c$ , where  $C_f = \{c : \text{dist}(s) < \alpha_c\}$ . Thus, we have

**COROLLARY A.2.** *The fractional weighted  $k$ -medians decision problem reduces to a mixed packing/covering problem of width  $\rho = k \max_f \{\sum_c \text{dist}(f, c)/d, 1\}$ . If a problem instance is feasible, the algorithm of [13] yields a fractional solution  $x$  with  $|x| \leq (1 + \epsilon)k$  and  $\text{dist}(x) \leq (1 + \epsilon)d$  in time linear in the input size times  $O(\rho \ln(m)/\epsilon^2)$ .*

If each  $\text{dist}(f, c) \leq d$ , then  $\rho \leq km$ , where  $m$  is the number of customers. This bound on the running time is a factor of  $m$  worse than the bound in Corollary 7.1 (though a reduction yielding smaller width may be possible).