

1. Sipser 7.22

SOLUTION SKETCH:

- (a) It suffices to observe that the property that 3 literals in any clause contains two with unequal truth values is preserved under negation of assignment.
- (b) Given a 3cnf-formula ϕ , define ϕ' to be the formula obtained by replacing each clause in ϕ by two clauses as described. Now, if ϕ is satisfiable, then in the formula ϕ' , fix the assignment to x_i in ϕ' to be that for x_i in ϕ , b to false, and z_i to true if y_1 and y_2 are false, and false otherwise. This shows that $\phi' \in \neq SAT$. Suppose on the other hand that $\phi' \in \neq SAT$. From (a), we may assume WLOG that b is assigned false in the \neq -assignment to ϕ' (otherwise, if b is true, just take the negation of the assignment). Then, fix the assignment to x_i in ϕ to be that for x_i in ϕ' . This would indeed be a satisfying assignment for ϕ . Otherwise, there is some clause in ϕ with y_1, y_2, y_3 assigned false, and thus z_i to true and b to true, a contradiction.
- (c) It is clear that $\neq SAT \in \mathbf{NP}$, and from (b), $3SAT \leq_P \neq SAT$. Hence, $\neq SAT$ is \mathbf{NP} -complete.

2. Sipser 7.23

SOLUTION SKETCH:

It is clear that $MAX-CUT \in \mathbf{NP}$. To see that $\neq SAT \leq_P MAX-CUT$, consider the reduction that on input 3-cnf formula ϕ , constructs the graph G_ϕ as described in the hint, and outputs $\langle \phi, n(3k)^2 + 2n \rangle$, where k is the number of clauses in ϕ , and n the number of variables in ϕ . If $\phi \in \neq SAT$, then the cut of G_ϕ that places all nodes corresponding to literals assigned true in one subset and the remaining nodes in the other has size $n(3k)^2 + 2n$.

Next, suppose $\langle \phi, n(3k)^2 + 2n \rangle \in MAX-CUT$. Take a maximal cut of G_ϕ , and assign the variables in one subset of the cut true, and the remaining variables false. Call this assignment a ; we will prove that a is well-defined. Observe that G_ϕ has $n(3k)^2 + 3n$, with $(3k)^2$ edges from the bipartite graph corresponding to each variable, and 3 edges from the triangle corresponding to each clause. Any cut of G_ϕ crosses the bipartite graph for each variable in at most $(3k)^2$ edges, and each triangle in at most 2 edges. Therefore, any cut of G_ϕ has size at most $n(3k)^2 + 2n$, with equality iff it crosses each bipartite graph for each variable in exactly $(3k)^2$ edges, and each triangle in exactly 2 edges. The former guarantees that the assignment a to the variables are consistent and therefore well-defined, and the latter guarantees that the assignment satisfies the local not-all-equal constraint on each clause.

3. Sipser 7.24

SOLUTION SKETCH:

It suffices to give a polynomial-time reduction from $SUBSET-SUM$ to D . On input $\langle S, t \rangle$, where $S = \{x_1, \dots, x_k\}$, output the polynomial in the variables a_1, \dots, a_k given by

$$p(a_1, \dots, a_k) = \left(t - \sum_{i=1}^k a_i x_i \right)^2 + \sum_{i=1}^k (a_i(1 - a_i))^2$$