SELECTED SOLUTIONS

5.5 Suppose for a contradiction that $A_{\mathsf{TM}} \leq_{\mathrm{m}} E_{\mathsf{TM}}$ via reduction f. It follows from the definition of mapping reducibility that $\overline{A_{\mathsf{TM}}} \leq_{\mathrm{m}} \overline{E_{\mathsf{TM}}}$ via the same reduction function f. However $\overline{E_{\mathsf{TM}}}$ is Turing-recognizable and $\overline{A_{\mathsf{TM}}}$ is not Turing-recognizable, contradicting Theorem 5.28.

5.6 Suppose $A \leq_{\mathrm{m}} B$ and $B \leq_{\mathrm{m}} C$. Then there are computable functions f and g such that $x \in A \iff f(x) \in B$ and $y \in B \iff g(y) \in C$. Consider the composition function $h(x) = g(f(x))$. We can build a TM that computes h as follows: First, simulate a TM for f (such a TM exists because we assumed that f is computable) on input x and call the output y. Then simulate a TM for g on y. The output is $h(x) = g(f(x))$. Therefore h is a computable function. Moreover, $x \in A \iff h(x) \in C$. Hence $A \leq_{\mathrm{m}} C$ via the reduction function h.

5.7 Suppose that $A \leq_{\mathrm{m}} \overline{A}$. Then $\overline{A} \leq_{\mathrm{m}} A$ via the same mapping reduction. Because A is Turing-recognizable, Theorem 5.28 implies that $\overline{A}$ is Turing-recognizable, and then Theorem 4.22 implies that A is decidable.

5.8 You need to handle the case where the head is at the leftmost tape call and attempts to move left. To do so add dominos

$$\left[\frac{\#qa}{\#rb}\right]$$

for every $q, r \in Q$ and $a, b \in \Gamma$, where $\delta(q, a) = (r, b, \mathrm{L})$.

5.10 Let $B = \{\langle M, w\rangle |\ M$ is a two-tape TM which writes a nonblank symbol on its second tape when it is run on $w\}$. Show that A_{TM} reduces to B. Assume for the sake of contradiction that TM R decides B. Then construct TM S that uses R to decide A_{TM}.

$S =$ "On input $\langle M, w\rangle$:
1. Use M to construct the following two-tape TM T:
   $T =$ "On input x:
   1. Simulate M on x using the first tape.
   2. If the simulation shows that M accepts, write a nonblank symbol on the second tape."
2. Run R on $\langle T, w\rangle$ to determine whether T on input w writes a nonblank symbol on its second tape.
3. If R accepts, M accepts w, therefore *accept*. Otherwise *reject*."

5.11 Let $C = \{\langle M\rangle |\ M$ is a two-tape TM which writes a nonblank symbol on its second tape when it is run on some input$\}$. Show that A_{TM} reduces to C. Assume for the sake of contradiction that TM R decides C. Construct TM S that uses R to decide A_{TM}.

$S =$ "On input $\langle M, w\rangle$:
1. Use M and w to construct the following two-tape TM T_w:
   $T_w =$ "On any input:
   1. Simulate M on w using the first tape.
   2. If the simulation shows that M accepts, write a nonblank symbol on the second tape."
2. Run R on $\langle T_w\rangle$ to determine whether T_w ever writes a nonblank symbol on its second tape.

3. If R accepts, M accepts w, therefore *accept*. Otherwise *reject*."

5.28 Assume for the sake of contradiction that P is a decidable language satisfying the properties and let R_P be a TM that decides P. We show how to decide A_{TM} using R_P by constructing TM S. First let $T_\emptyset$ be a TM that always rejects, so $L(T_\emptyset) = \emptyset$. You may assume that $\langle T_\emptyset \rangle \notin P$ without loss of generality, because you could proceed with $\overline{P}$ instead of P if $\langle T_\emptyset \rangle \in P$. Because P is not trivial, there exists a TM T with $\langle T \rangle \in P$. Design S to decide A_{TM} using R_P's ability to distinguish between $T_\emptyset$ and T.

$S =$ "On input $\langle M, w \rangle$:

1. Use M and w to construct the following TM M_w:
$M_w =$ "On input x:
1. Simulate M on w. If it halts and rejects, *reject*.
If it accepts, proceed to stage 2.
2. Simulate T on x. If it accepts, *accept*."

2. Use TM R_P to determine whether $\langle M_w \rangle \in P$. If YES, *accept*.
If NO, *reject*."

TM M_w simulates T if M accepts w. Hence $L(M_w)$ equals $L(T)$ if M accepts w and $\emptyset$ otherwise. Therefore $\langle M, w \rangle \in P$ iff M accepts w.

5.30 **(a)** *INFINITE*$_{\mathsf{TM}}$ is a language of TM descriptions. It satisfies the two conditions of Rice's theorem. First, it is nontrivial because some TMs have infinite languages and others do not. Second, it depends only on the language. If two TMs recognize the same language, either both have descriptions in *INFINITE*$_{\mathsf{TM}}$ or neither do. Consequently, Rice's theorem implies that *INFINITE*$_{\mathsf{TM}}$ is undecidable.