

Problem 1: Prove or disprove: *the following language is decidable:*

$L = \{ \langle M, n \rangle : M \text{ is a Turing machine with } n \text{ states, and there exists a string } w \text{ in } \Sigma^* \text{ of length at most } n^2 \text{ such that } M \text{ accepts } w \text{ in at most } n^3 \text{ transitions} \}$

The language is decidable. The following Turing machine decides it:

$M_L =$ “On input $\langle M, n \rangle$

1. Check if M has n states. If not, reject.
2. For each string $w \in \Sigma^*$ of length at most n^2 do:
3. Simulate M on w for n^3 steps.
4. If M accepts w within n^3 steps, accept.
5. Reject.

Problem 2: Prove or disprove: *the following language is decidable:*

$L = \{ \langle M, p, q \rangle : \text{TM } M, \text{ on input '1', visits state } p \text{ (at some point) and (at a later point) state } q \}$.

The language L is not decidable. To prove it, we show that A_{TM} reduces to L . The Turing machine that computes the reduction is the following:

$M_R =$ “On input $\langle M, w \rangle$:

1. Construct Turing machine M' :

$M' =$ “On input x :

1. Simulate M on w . If M accepts, accept.”
2. Let p and q be, respectively, the start and accept states of M' .
3. Output $\langle M', p, q \rangle$.”

Clearly M_R always halts. To see that M_R reduces A_{TM} to L :

$$\begin{aligned} \langle M, w \rangle \in A_{TM} & \\ \Leftrightarrow M \text{ accepts } w & \\ \Leftrightarrow M' \text{ accepts } 1 & \\ \Leftrightarrow M' \text{ on input } 1, \text{ enters the start state and, later, the accept state} & \\ \Leftrightarrow \langle M', p, q \rangle \in L & \end{aligned}$$

Problem 3: A *linear-bounded automaton (LBA)* is a restricted form of Turing machine wherein the tape head isn't permitted to move off the portion of the tape containing the input. If the machine tries to its head off either either end of the input, the head stays where it is. Define

$$NE_{LBA} = \{ \langle M \rangle : M \text{ is an LBA and } L(M) \neq \emptyset \}$$

Consider the following “proof” that NE_{LBA} is not decidable:

Given a TM M and input w , define $f(\langle M, w \rangle) = \langle B \rangle$ where B is the following LBA

$B =$ “On input 0^n :

1. If $n < 100|w|$ reject.
2. Otherwise, simulate M on w for n steps.
3. If M accepts w within n steps, accept.”

The function f is a reduction from A_{TM} to NE_{LBA} . That is, f is computable and $\langle M, w \rangle \in A_{TM} \Leftrightarrow \langle B \rangle \in NE_{LBA}$. To verify the latter property, note that

$$\begin{aligned} \langle M, w \rangle \in A_{TM} & \\ \Leftrightarrow \exists n \text{ such that } M \text{ accepts } w \text{ within } n \text{ steps} & \\ \Leftrightarrow \exists n \text{ such that } B \text{ accepts } 0^n & \\ \Leftrightarrow \langle B \rangle \in NE_{LBA} & \end{aligned}$$

Therefore A_{TM} reduces to NE_{LBA} . Since A_{TM} is not decidable, we conclude that NE_{LBA} is not decidable.

Is the proof correct? If not, what is wrong with it?

The proof is correct.