CS215 ASSIGNMENT 4

due Tuesday, Nov. 16, 8AM

For each problem, you are to give a complete proof that the problem is NP-complete:

Problem 1.

2CLIQUE:

Instance: An undirected graph G, positive integer K;

Query: Does G have two disjoint cliques each of size K?

We give a polynomial-time reduction from CLIQUE to 2CLIQUE:

Given a graph G and integer k, construct graph G' by making two disjoint copies of G, then taking G' to be their union.

More formally, if $G = (V, E)$, take $G' = (V', E')$ where $V' = \{[i, v] : i \in \{0, 1\}, v \in V\}$ and $E' = \{([i, v], [i, w]) : i \in \{0, 1\}, (v, w) \in E\}$.

The reduction is then $f(G, k) = \langle G', k\rangle$.

The reduction is clearly polynomial time.

To prove that it is correct, we show that G has a clique of size k if and only if G' has two disjoint cliques of size k. This is fairly obvious, but we'll prove it anyway.

($\Rightarrow$) *Suppose G has a clique C of size k. Then for $i = 1, 2$ let $C_i = \{[i, v] : v \in C\}$. Then C_1 and C_2 are disjoint, and each is a clique in G'. So G' has two disjoint cliques of size k.*

($\Leftarrow$) *Suppose G' has two disjoint cliques C_1 and C_2 of size k. Consider the clique C_1. Since there are no edges of the form $([0, v], [1, w])$ in G', either all of the vertices of C_1 are of the form $[0, v]$ or they are all of the form $[1, v]$. That is, C_1 is contained entirely in one of the two copies of G. Define $C = \{v : [0, v] \in C_1 \vee [1, v] \in C_1\}$. Then C is a clique in G (prove it if you like), and C has size k.*

Problem 2.

VISIT-EDGES:

Instance: An undirected graph $G = (V, E)$, set of edges $F \subseteq E$;

Query: Is there a cycle in G that traverses each edge in F?

Note: above we mean a *simple* cycle, that is a cycle that does not visit the same vertex twice.

We give a reduction from HAMILTONIAN CYCLE. Given a graph $G = (V, E)$, construct the bipartite graph $G' = (V', E')$, where:

$$\begin{aligned} V' &= \{[i, v] : i \in \{0, 1\}, v \in V\} \\ E' &= A \cup B, \text{ where} \\ A &= \{([i, v], [j, w]) : (v, w) \in E, i \neq j\}, \\ B &= \{([0, v], [1, v]) : v \in V\}. \end{aligned}$$

In words, we are making two copies $[0, v]$ and $[1, v]$ of every vertex v in the original graph. We add an edge between each pair of vertices like that, then for each edge (u, w) in the original graph we add two edges $([0, u], [1, w])$ and $([0, w], [1, u])$.

The reduction is $f(G) = \langle G', B$.

Clearly the reduction is polynomial time. Next we prove it is correct — that G has a simple cycle that visits all of its vertices if and only if G' has a simple cycle that crosses every edge in A.

($\Rightarrow$) Suppose G has a simple cycle C that visits all its vertices. Name the vertices along cycle C as $v_1, v_2, \ldots, v_n, v_1$ in the order they are visited by C.

Define cycle C' to visit the vertices (in G') as follows:

$[0, v_1], [1, v_1], [0, v_2], [1, v_2], [0, v_3], [1, v_3], \ldots, [0, v_n], [1, v_n], [0, v_1]$.

Then, because C is a simple cycle in G that visits all the vertices in V, C' is a simple cycle in G' (verify this!) that visits all the edges in A.

($\Leftarrow$) Suppose G' has a simple cycle C' that uses all the edges in A. Since every vertex in G' is on an edge in A, the cycle C' must be a Hamiltonian cycle in G'. Since each vertex in G' touches exactly one edge in A, the edges traversed by the cycle C' must alternate between edges in A and edges not in A (verify!). Also, all of the edges in E' connect a vertex $[i, v]$ to a vertex $[j, w]$ where $i \neq j$. Thus, vertices C' visits, in order, can be named as

$$[0, v_1], [1, v_1], [0, v_2], [1, v_2], \ldots, [0, v_n], [1, v_n], [0, v_1]$$

where the v_i's are distinct and, for each i, $([0, v_i], [1, v_i])$ is an edge in A and (v_i, v_{i+1}) is an edge in E (the original graph). Thus, the cycle $C = (v_1, v_2, \ldots, v_n, v_1)$ is a Hamiltonian cycle in G.

Problem 3.

SET-SPLITTING:

Instance: A finite set S and a collection C of finite subsets of S;

Query: Can the elements of S be colored with two colors, say red and green, so that no set $X \in C$ has all elements colored with the same color?

As an example, suppose that $S = \{1, 2, 3, 4, 5, 6\}$ and

$$C = \{\{1, 2\}, \{3, 4, 5\}, \{2, 3, 6\}, \{1, 4, 6\}, \{2, 5\}\}.$$

If $1, 3, 5$ are green, and $2, 4, 6$ are red, then each set in C has elements of two different colors.

In the following instance:

$$C = \{\{1, 2\}, \{3, 4, 5\}, \{2, 3, 6\}, \{1, 4\}, \{2, 5\}, \{1, 3\}, \{5, 6\}\},$$

there is no good coloring (why?).

To prove SET-SPLITTING is NP-Complete, we need to show (1) SET-SPLITTING is in NP and (2) SET-SPLITTING is NP-Hard.

To see (1) is easy because, given a coloring, it is easy to verify in poly time that no set is monochromatic.

To prove that SET-SPLITTING is NP-hard, we reduce 3SAT to it.

Let ϕ be a 3CNF formula with variable set V.

Construct the following instance of SET-SPLITTING. The set of elements S is $\{F\} \cup V \cup \{\bar{X} : X \in V\}$. Here F is a new element not related to any variable. Each other element corresponds to a variable or its negation.

Build the collection of sets C as follows. For each variable X in ϕ, construct a set $S_X = \{X, \bar{X}\}$. For each clause c (e.g. $X \vee Y \vee \bar{Z}$), construct a set $S_c = \{X, Y, \bar{Z}, F\}$. Here F is a new element not related to any variable. (There is only one such element F — it is the same across all sets built for clauses.)

So, the reduction is $f(\phi) = (S, C)$ where S and C are as described above.

Clearly the reduction is polynomial time.

To finish we prove that ϕ is satisfiable if and only if (S, C) can be colored so that no set is monochromatic.

($\Rightarrow$) Suppose ϕ is satisfiable. Fix some satisfying assignment.

Consider the following coloring of the elements in S. Color the element F 'red'. For each variable X that is assigned 'false', color the elements X and $\bar{X}$ 'red' and 'green', respectively. For each variable X that is assigned 'true', color the elements X and $\bar{X}$ 'green' and 'red', respectively.

As long as the assignment was satisfying, this coloring makes no set monochromatic. For each variable X, the set $S_X = \{X, \bar{X}\}$ has one red and one green element. For each clause c, the set S_c has at least one red element (F) and, because some literal in the clause has a value of true, S_c has at least one 'green' element.

Thus, $(S, C) \in$SET-SPLITTING.

($\Leftarrow$) Suppose $(S, C) \in$SET-SPLITTING. Fix some coloring of S with two colors such that every set has at least one element of both colors.

Consider the following assignment to the variables of ϕ. For each variable X, assign it 'true' if its color differs from that of the element F. Assign X 'false' if its color is the same as that of the element F.

Then each clause c in ϕ is satisfied, because the set S_c has at least one element X or $\bar{X}$ that is colored differently than F.

Thus, $\phi \in$SAT.

Collection: I will collect the homeworks in class on Tuesday. If you can't turn it in in class, slip it under my office door (by no later than 8AM).