Simpler multi-threaded model checking via new foundations for implicit encodings

C++ template metaprogramming rocks!

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Abstract

This research proposes to improve the performance of model checking for asynchronous systems, and the quality and speed of model checking research coding, by orders of magnitude, using the advantages of parallel processing, well defined encodings (GDDs) with desirable properties, enabling clean high-level interfaces to those encodings.

Symbolic model checking employs decision diagrams (DDs) to encode large models, and libraries, such as TeDDy, to manipulate them. I propose an enhanced interface to TeDDy, permitting users to create novel efficient model-checking algorithms without resorting to library modification or writing code to manipulate concrete data structures. The novel library interface, defined through C++ metaprogramming, will allow the writing of C++ algorithm code in a significantly less cluttered style, while automatically generating code having the performance of existing hand-optimized DD code and the compactness of fully-identity-reduced DDs (FIDDs).

The needs, for consistency in the desired interface, and for compactness of FIDDs (for encoding asynchronous transition relations), in turn motivated the invention of (Generalized Decision) Diagrams (GDDs). GDDs are a novel form of DD which retains the compactness of FIDDs, while providing closure over all DD operations desired for consistency of the TeDDy interface (FIDDs are not closed). I have recently completed proof of the necessary properties of GDDs. As a proof-of-concept demonstration of the efficiency and convenience of the novel interface, I propose to measure the simplification in code of my existing bisimulation algorithms adapted to use the new TeDDY, expecting significant reduction, while having the same, or improved, efficiency compared to existing code.

I additionally propose to extend my existing fully-symbolic bisimulation research in the following two directions: (1) I will explore using 'uncertain' transition relations to augment a transition system so as to effectively improve the overall locality of transition relations, to improve my saturation-based weak bisimulation algorithm, and (2) I will study a new fully-symbolic lumping algorithm that uses the saturation heuristic, in a manner similar to our fully-symbolic weak bisimulation algorithm. These projects will illustrate the enhancement in the quality and rate of research that results from the use of improved tools and interfaces.

I additionally propose to explore using novel speculation heuristics in implementing a scalable parallel version of TeDDy on a multi-threaded parallel execution platform. My novel speculation heuristics may give, to decision diagram manipulation code, the ability to obtain significant parallel speedups without user code modification. To avoid user code modification, I must also define a parallel version of the improved library interface, as the C++ metaprogramming involves both high- and low-level aspects of the TeDDy interface. The resulting GDD/TeDDy system will support convenient creation of multi-threaded parallel decision diagram manipulation software.

Saturation, currently the fastest heuristic for exploring finite state spaces of asynchronous systems, is critical to model checking of asynchronous finite transition systems. The recursive and efficiently lazy nature of saturation has restrained attempts to obtain significant scalable parallel speed-ups in parallel implementations. Parallel saturation implementations typically obtain some limited speed-up due to their ability to access the main memory of multiple processors, decreasing the occurrence of page thrashing. I propose to re-organize parallel saturation, by inserting additional novel structural recursion that is organized according to both processor locality and model locality. The resulting system supports convenient construction of multi-threaded parallel model checking software.

I propose demonstrating the value of the resulting system by improving the performance of an application.

1 Introduction

DDs compactly encode large sets and relations used in model checking and functions used in stochastic model checking. This research aims to produce an improved library for DDs by implementing a novel form of DD that is conducive to our purpose, and to illustrate its effectiveness on some model checking tasks.

I briefly summarize the state of the practice with respect to sequential DD programming and parallel DD programming, describing potential advances with my approach.

1.1 Decision Diagrams

As originally described, a *binary decision diagram* (BDD) is a directed graph or shared binary tree used to encode a function of multiple (K) binary variables (a K-tuple of booleans), where the range is also boolean [6](§6.37). In the worst case, a BDD is a binary decision tree and occupies space proportional to the size of the table of the function encoded. In some practical cases, many subtrees of a decision tree are identical, so the BDD encoding occupies considerably less space. A binary decision tree is a constant-depth binary tree with the depth being the number of variables input to the encoded function, and the leaves being boolean range values. We do not yet consider *reduced* DDs, which do not have constant depth. Each node of the tree has a *level*, which is the distance from the node to a leaf. Each level of the tree corresponds to one of the input variables, according to the *variable ordering*. Each node of the tree has 2 outgoing edges, labeled 0, and 1, corresponding to the values of the input variable associated with the level of that node.

An encoded function f_A (of type: $\mathbb{B}^K \to \mathbb{B}$) is evaluated for its tuple X ($\in \mathbb{B}^K$) of K boolean arguments $X_1, \ldots, X_K$ using its decision tree or BDD encoding A as follows:

1. let b refer to the root of A
2. for each $i \in K, \ldots, 1$, in decreasing order

   if X_i is true let b refer to the node from b reached by following the edge labeled 1,

   otherwise let b refer to the node from b reached by following the edge labeled 0.
3. b now refers to a leaf which is the result of the function.

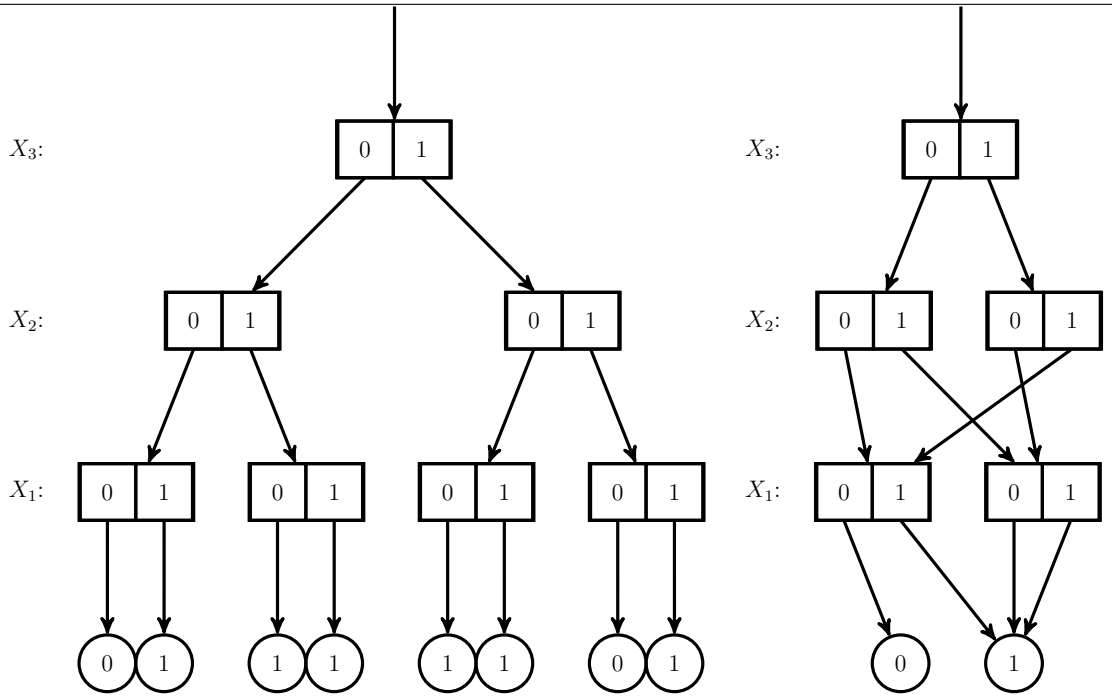
When illustrating decision diagrams, the edge labels are usually indicated in the body of the node from which their corresponding edges proceed.

Fig. 1 shows both encodings of the function $f(X) = (X_3 \oplus X_2) \vee X_1$, illustrating that the BDD encoding is more compact (requires fewer nodes).

These illustrations also provide a hint that decision trees resemble finite automata accepting languages consisting of a set of finite strings of fixed length, and the leaves correspond to terminal nodes and are additionally labeled with range values. Decision diagrams then correspond to minimized automata.

Decision diagrams are thus a form of compressed representation of their encoded functions, and thus may not always be more compact than an explicit representation. To efficiently detect sharable sub-trees, it helps for an encoding to be *canonical*. A canonical encoding is one that has at most a single encoding for a given encoded function, so that there are no two encodings for the same function. With a canonical encoding, no function is ever encoded redundantly. With many simple forms of decision diagram, canonicity is easy to prove. Many complex forms of decision diagram have been proposed, which in many cases are not canonical. In this proposal, my novel DD encoding is proved canonical, and so supports efficient sharing of sub-trees.

The performance of algorithms on decision diagrams benefits from the structure and compactness of their encoding. New trees/nodes are constructed by the *unique*(...) function, which receives as input an



(a.) Decision tree encoding (b.) BDD encoding

Fig. 1: Encodings of $f(X) = (X_3 \oplus X_2) \vee X_1$

ordered collection of nodes for use as the child nodes of the newly constructed node. If an existing node already has the same child nodes in the same order, the existing node is returned instead, invisibly implementing sharing among sub-trees. Fundamental algorithms on decision diagrams are typically recursive tree-style traversals that depend for efficiency on memoization of function calls. For illustration, I will give a BDD algorithm for union of two sets. Sets are represented as encoded characteristic functions, so the characteristic function $f_{a\cup b}$ of the result $a \cup b$ of the union of a and b represented as their characteristic functions f_a and f_b, respectively, is $f_{a\cup b}(x) = f_a(x) \vee f_b(x)$, that is, simply the vector 'or' operation on the individual components of the encodings of f_a and f_b. The BDD union algorithm is simply an adaptation of this definition applied to a tree-like encoding of such an array.

This BDD algorithm for union of two sets (a and b of K-tuples) is given here:

memoized Function $union(a, b, K) : BDD \times BDD \times \mathbb{N} \to BDD$ is:

if $(K = 0)$ then: $a, b \in \mathbb{B}$

$union(a, b, K) = a \vee b$

else $(K > 0)$

Variable *children* is Array[$\mathbb{B}$] of BDD An array of 2 BDDs

Variable $c : BDD$

For all $X_K \in \mathbb{B}$ Loop over possible values of X_K.

Let $a', b' =$ child X_K of a, child X_K of b

Assign $children[X_K] \leftarrow union(a', b', K - 1)$ recursive subdivision for part X_K

Assign $c \leftarrow unique(children)$

$union(a, b, K) = c$

Complement of sets represented by their encoded characteristic functions uses the formula $f_{\bar{s}}(X) = \neg f_s(X)$, and can be easily implemented in BDD libraries as an operation which (modulo canonicity) copies a BDD, and when reaching the leaves, substitutes '1' for '0', and '0' for '1'.

Thus BDD-encoded sets are closed under union and complement, and obviously all other set operations, such as conjunction, disjoint union etc.

Many additional refinements are employed in practical versions of the above algorithm.

Parallel processing has been employed with some success for the processing of DDs used in model checking. Thus far, parallel processing has allowed model checking of larger models than possible with sequential processing, but has not produced hoped-for massive improvements in speed of model checking. This research provides another opportunity to attempt an improvement in the speed-up provided by parallel model checking.

1.2 Structure of this Proposal

Section 2 includes greater detail about some of the great variety of decision diagram types that have been studied since the origin of BDDs, as well as discussion of implementations and applications of decision diagrams relevant to this proposal. Section 3 explains GDDs, my novel form of decision diagram and my proposal for implementing and applying them while Section 5 provides detailed information on the intended progress of this research.

2 Background

2.1 Multi-Way Decision Diagrams

Multi-way decision diagrams (MDDs) generalise BDDs by expansion of the types of the members of the tuple domain, so that the domain is $D_K \times \ldots \times D_1$, where, for each $k \in 1, \ldots, K$, D_k is any chosen finite set, assumed to be $\{0, \ldots, Max_k\}$ (where each $Max_k \in \mathbb{N}$), without loss of generality, for a suitable selection of $Max_K, \ldots, Max_1$.

Thus MDDs encode all functions of type $\{0, \ldots, Max_K\} \times \ldots \times \{0, \ldots, Max_1\} \to \mathbb{B}$.

Like BDDs, MDDs are directed graphs resulting from sharing nodes in a tree, where the tree is similar to an automata for accepting a language of strings of fixed length. An MDD is identical to a BDD when $Max_k = 1$ for all $k \in K \ldots 1$. When $Max_k > 1$ for some k, nodes at level k have $Max_k + 1$ outgoing edges, each labeled with a distinct member of $\{0, \ldots, Max_k\}$. As with BDDs, drawings of MDDs show edge labels in the body of the node from which the edge proceeds. Inasmuch as tuples of any finite type can be encoded as boolean tuples, it is of course possible to encode functions of type $\{0, \ldots, Max_K\} \times \ldots \times \{0, \ldots, Max_1\} \to \mathbb{B}$ as BDDs instead of MDDs, however there are many cases where it is less natural to do so, and where the MDD encoding of such functions is more efficient to use.

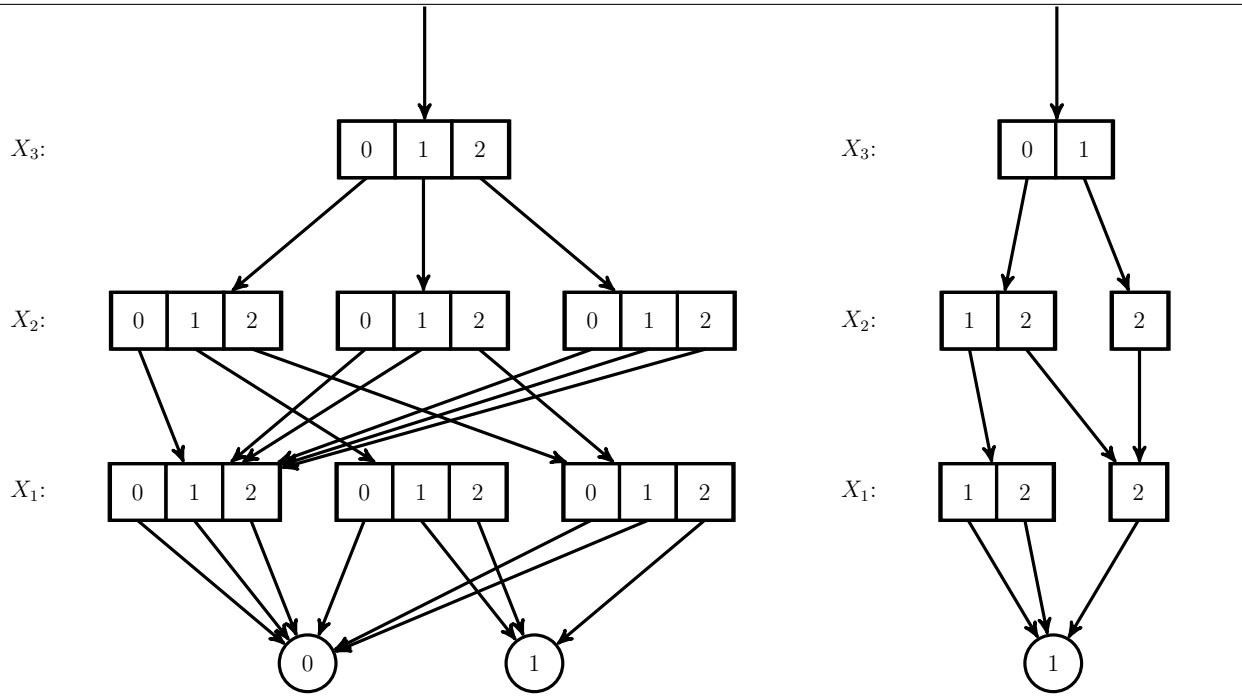
Fig. 2.a shows an MDD encoding the function $f(X) = (X_3 < X_2 \leq X_1)$ where $X_1, X_2, X_3 \in \{0, \ldots, 2\}$. A commonly used improvement, called *null pointer elimination* to MDDs removes all edges that always lead to the '0' leaf, as well as nodes having only such edges. Fig. 2.b illustrates the memory savings resulting from null pointer elimination applied to an MDD for the same function.

Practical MDD implementations employ many additional encoding devices to further reduce memory usage.

Here I give an algorithm for evaluation of an MDD-encoded function (with null pointer elimination):

An encoded function f_A (of type: $\{0, \ldots, Max_K\} \times \ldots \times \{0, \ldots, Max_1\} \to \mathbb{B}$) is evaluated for its tuple X ($\in \mathbb{N}^K$) of K arguments $X_1, \ldots, X_K$ using its MDD encoding A as follows:

1. let b refer to the root of A



(a.) Without null pointer elimination (b.) With null pointer elimination

Fig. 2: MDDs encoding the function $f(X) = (X_3 < X_2 \leq X_1)$ given $X_1, X_2, X_3 \in \{0, \ldots, 2\}$

2. for each $i \in K, \ldots, 1$, in decreasing order

   if b has an outgoing edge labeled with the value of X_i, let b refer to the node from b reached by following the edge labeled with the value of X_i.

   otherwise the value of the function is $false$, hence terminate early.

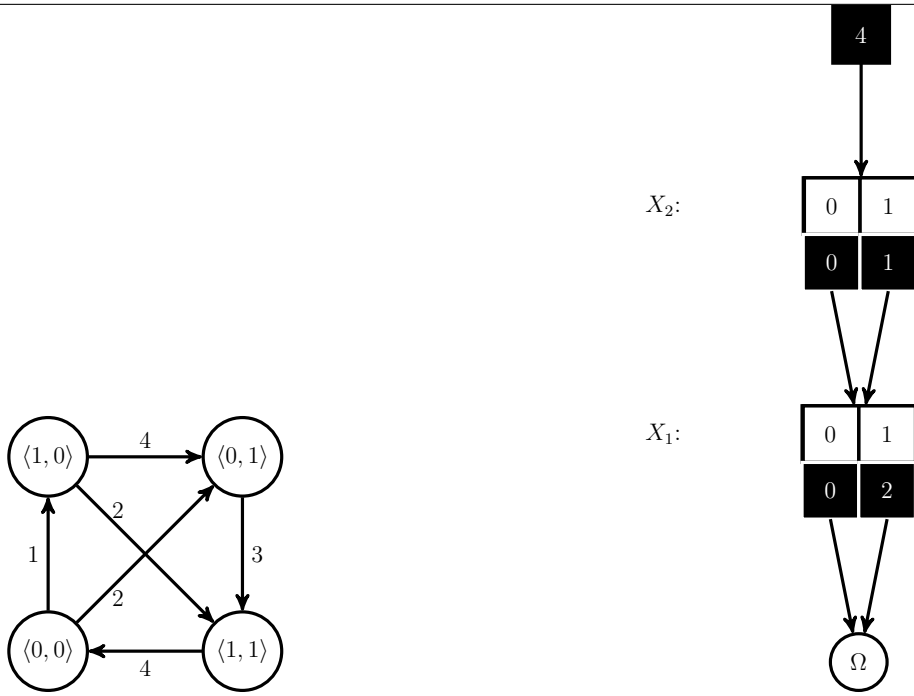
3. b now refers to a leaf which is the result of the function.

2.1.1 Edge-valued Multi-Way Decision Diagrams

The possible range of encoded functions may be extended beyond the booleans $\mathbb{B}$ to a larger range R by simply employing additional values in R for leaf nodes. Such MDDs are called *Multi-Terminal Multi-way Decision Diagrams*, or MTMDDs. MTMDDs are most efficient in certain cases where the range is relatively small compared to the domain. In many other cases, *Edge-Valued Multi-way Decision Diagrams* (EVMDDs) are more compact by many orders of magnitude, and allow the use of ranges that are not explicitly known a-priori.

Each edge, including the root incoming edge, of an EVMDD stores an operation, encoded as an *edge value*. To evaluate the encoded function, an 'accumulator' variable is initialized with a default value, and modified by each operator encountered during traversal of the EVMDD, leaving the function result in the 'accumulator' variable. No specific value is associated with leaves, so only one leaf exists, conventionally labeled Ω.

An EV+MDD is an EVMDD where the default value is 0, and the operation stored at each edge is the addition of the edge value which is in $\mathbb{N}$, while an EV*MDD has a default value of 1, and the operation is multiplication by the edge value. EV*MDDs have been used to compactly encode transition probabilities in Markov process graphs [52](§6.56). EV+MDDs have been used to compactly encode distance information



(a.) Directed graph with weighted edges (b.) EV+MDD encoding of distance from $\langle 0, 0\rangle$

Fig. 3: EV+MDD encodes distance in directed graph

for transition graphs [12](§6.50). Fig. 3 shows (a.) a transition graph, and (b.) an EV+MDD encoding of the distance (from a specific node) for all the nodes in the graph. Note that the edge value for each edge is customarily displayed in a black box at the top of the corresponding drawn edge. Canonicity of EVMDDs is more complex than the case for MDDs, and will not be explained here.

To clarify, the algorithm for evaluation of EV+MDD-encoded functions is given here:

An encoded function f_A (of type: $\{0, \ldots, Max_K\} \times \ldots \times \{0, \ldots, Max_1\} \to \mathbb{N} \cup \{\infty\}$) is evaluated for its tuple X ($\in \mathbb{N}^K$) of K arguments $X_1, \ldots, X_K$ using its EV+MDD encoding A as follows:

1. let b refer to the edge leading to the root of A
2. let v be the edge value of the edge b
3. initialize $r \leftarrow v$
4. for each $i \in K, \ldots, 1$, in decreasing order

   if the target of b has an outgoing edge labeled with the value of X_i, let b refer to the edge labeled with the value of X_i.

   otherwise the value of the function is ∞, hence terminate early.

   let v be the edge value of the edge b

   calculate $r \leftarrow r + v$
5. Now r is the result of the function.

2.1.2 Extensible Multi-Way Decision Diagrams

Extensible mult-way decision diagrams (EMDDs) provide encoded functions with limited access to infinite domains. MDDs with K levels encode certain functions of type: $\mathbb{N}^K \to R$ (for some range R).

The encoding is different from plain MDD encoding in that an EMDD node may have any finite number of outgoing edges, and they are (uniquely) labeled with naturals, except there is always one outgoing edge labeled as '*'. The algorithm for evaluating an encoded function f_A (of type: $\mathbb{N}^K \to R$) on its argument X ($\subseteq \mathbb{N}^K$) using its EMDD encoding A is altered to the following:

1. let b refer to the root of A
2. for each $i \in K, \ldots, 1$, in decreasing order

   let b refer to the node reached from an edge labeled with the value of X_K (if such an edge from b exists)

   let b refer to the node reached from the edge labeled '*' (otherwise)
3. b now refers to a leaf which is the result of the function.

Typical implementations may impose some additional restrictions on the collection of labels of edges going out from a node. TeDDy requires that the natural labels be in a contiguous range starting from 0, and, to enforce canonicity, the edge having the largest natural label must not lead to the same node as the edge labeled '*'.

Note that without the contiguity restriction, enforcing canonicity would require that from a given node, the edge labeled '*' must not lead to the same node as any other edge from the given node.

The '*unique*' function is more complex than in the BDD case, as it must discard certain redundant children to enforce canonicity.

The EMDD encoding allows some infinite sets (along with all finite sets) of natural tuples to be represented via their encoded characteristic functions. The complement of such sets may be taken using the same algorithm as for BDD-encoded sets, however the union algorithm for EMDD-encoded sets is more complex, and is given here:

memoized Function $union(a, b, K) : EMDD \times EMDD \times \mathbb{N} \to EMDD$ is:

if $(K = 0)$ then: (leaf case where $a, b \in R$)

$union(a, b, K) = a \vee b$

else $(K > 0)$

Let s be the set of all labels of edges from a or from b

Variable *children* is Map[s] to $EMDD$ (A mapping of edge labels to EMDDs)

Variable $c : EMDD$

For all $X_K \in s$ (Loop over possible values of X_K (or '*'))

Let $a', b' =$ child X_K of a, child X_K of b (use child '*' when child X_K does not exist)

Assign $children[X_K] \leftarrow union(a', b', K-1)$ recursive subdivision for part X_K

Assign $c \leftarrow unique(children)$

$union(a, b, K) = c$

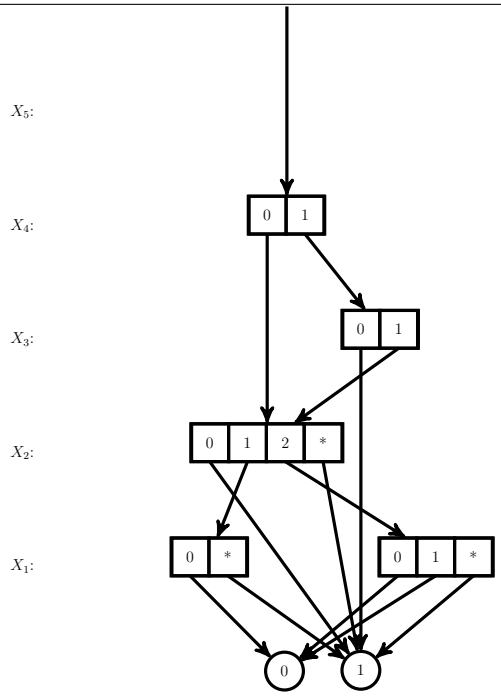


Fig. 4: Fully-Reduced EMDD encoding $f(X) = (X_4 \wedge \neg X_3) \vee (X_2 \leq X_1) \vee (X_2 > 2)$ over $\mathbb{N} \times \mathbb{B} \times \mathbb{B} \times \mathbb{N} \times \mathbb{N}$

Of course, practical implementations of the above algorithm employ a great many additional refinements.

EMDDs (and EVMDDs, MDDs, and BDDs) can all be used in combination with the following reductions, which, in many important cases, can produce many orders of magnitude of additional improvement in storage space and computation time. Until now, all the types of DD we have seen have edges only between adjacent levels. Thus the level of a node can easily be determined by subtracting 1 from the level of its predecessor. Reductions assign meaning to diagrams where an edge may *skip* one or more levels, by leading from a node at some level k, to a node at some level $j < k - 1$. In these cases, an explicit level number may be stored with each node, so that its level number may be easily determined in the presence of edges that skip levels. Within algorithms, a *level*() function mapping nodes to $\mathbb{N}$ is employed to give the level of any node.

2.1.3 Fully-Reduced Multi-Way Decision Diagrams

Fully-Reduced MDDs utilize implicit knowledge of the domain when assigning meaning to diagrams where level(s) are skipped. An edge e from a node A at level $k + 1$ which skips level k and leads to a node C at some level $j < k$ is a shorthand for the case where the edge e instead leads to a node B at level k, and where node B has $|D_k|$ children, each uniquely labeled with a member of D_k, and each leading to C (When $D_k = \mathbb{N}$, the shorthand is slightly different). Hence, the output of the encoded function is independent of X_k. Thus, Fully-Reduced MDDs provide a compact way to encode a function that, on some path, ignores a member X_k of its input tuple X, leading to considerable reduction of space when functions frequently ignore many members of their input tuples. Fig. 4 shows a Fully-Reduced MDD with null-pointer elimination that encodes $f(X) = (X_4 \wedge \neg X_3) \vee (X_2 \leq X_1) \vee (X_2 > 2)$ over $\mathbb{N} \times \mathbb{B} \times \mathbb{B} \times \mathbb{N} \times \mathbb{N}$.

The algorithm for evaluating an encoded function f_A on its argument X using its Fully-Reduced MDD (or Fully-Reduced EMDD) (and allowing null-pointer elimination) encoding A is altered to the following:

1. let b refer to the root of A (or the unique leaf of A, if all levels are skipped)
2. for each $i \in K, \ldots, 1$, in decreasing order

   if $i = level(b)$

   let b refer to the node reached from an edge labeled with the value of X_i (if such an edge from b exists)

   otherwise:

   let b refer to the node reached from the edge labeled '*' (if such an edge from b exists)

   otherwise the value of the function is $false$, hence terminate early.
3. b now refers to a leaf which is the result of the function.

Note that this algorithm is an extension of the corresponding algorithms for both EMDDs and for MDDs with null-pointer elimination. As the complexity of the decision diagram types increases, the conditions for the canonicity of a decision diagram become more complex. For the sake of brevity I will not discuss the canonicity conditions for diagrams with reductions.

2.1.4 Identity-Reduced Multi-Way Decision Diagrams

Identity-Reduced diagrams provide space reduction when encoding a function that sometimes depends on whether or not some tuple member X_k is equal to the preceding tuple member X_{k+1}. In those times, such a function returns false when $X_k \neq X_{k+1}$. In this case, an edge that skips from node A at level $k+1$ to some node C at level $j < k$ indicates, for tuples that invoke this path, that the value of the function is $false$ if $X_{k+1} \neq X_k$. Identity reduction is generally not used alone, but in combination with full reduction as discussed next.

2.1.5 Fully-Identity-Reduced Multi-Way Decision Diagrams

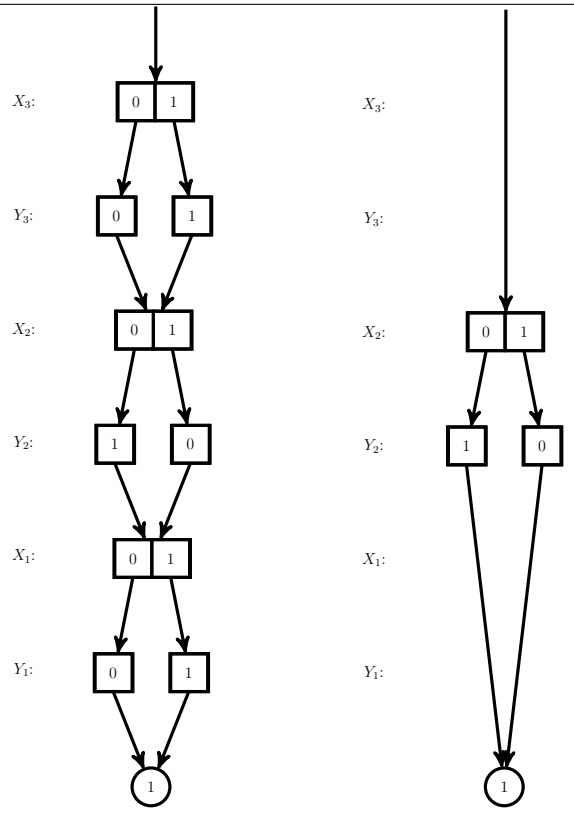
Fully-Identity-Reduced MDDs incorporate 'full' reduction at even levels and 'identity' reduction at odd levels. A 2-level Fully-Identity-Reduced MDD, where both levels are skipped, and the edge leads to the *true* node, compactly encodes the identity relation between X_2 and X_1, independently of the (necessarily common) domain of X_2 and X_1. This so-called identity pattern may be used to advantage in multiple parts of a Fully-Identity-Reduced MDD. Doing so has been found to greatly reduce space requirements for interleaved encodings of transition relations where the *support* of the transition relation omits many variables. The support of a transition relation is the set of variables used in or changed by the transition relation. Thus it has found application in practical model checking systems. These advantages must not be ignored when considering encodings for such transition relations, and my novel structures retain these advantages in certain variants.

The algorithm for evaluating an encoded function f_A on its argument X using its Fully-Identity-Reduced MDD (or Fully-Identity-Reduced EMDD) (and allowing null-pointer elimination) encoding A is altered to the following:

1. let b refer to the root of A (or the unique leaf of A, if all levels are skipped)
2. for each even $i \in \{K, K-2, \ldots, 2\}$, in decreasing order

   (process an even (Fully-Reduced) level):

   if $i = level(b)$ (the Fully-Reduced level is not skipped)



(a.) Fully-Reduced (b.) Fully-Identity-Reduced
(Both with null-pointer elimination)

Fig. 5: Fully-Identity-Reduced MDDs encoding $f(X) = (\langle Y_3, Y_2, Y_1\rangle = \langle X_3, \neg X_2, X_1\rangle)$ over $\mathbb{B}^{2^3}$. Note that all node levels in the Fully-Identity-Reduced encoding are in the support $(\{X_2, Y_2\})$ of f

let b refer to the node reached from an edge labeled with the value of X_i (if such an edge from b exists)
otherwise:
let b refer to the node reached from the edge labeled '*' (if such an edge from b exists)
otherwise the value of the function is $false$, hence terminate early.

(process the odd (Identity-Reduced) level just below):
if $i - 1 = level(b)$ (the Identity-Reduced level is not skipped)
let b refer to the node reached from an edge labeled with the value of X_{i-1} (if such an edge from b exists)
otherwise:
let b refer to the node reached from the edge labeled '*' (if such an edge from b exists)
otherwise the value of the function is $false$, hence terminate early
otherwise (the Identity-Reduced level is skipped)
if $X_i \neq X_{i-1}$, then the value of the function is $false$, hence terminate early

3. b now refers to a leaf which is the result of the function.

Fig. 5 illustrates the improvement gained using Fully-Identity-Reduction for encoding interleaved relations having a support that is small compared to the total number of variables.

2.1.6 GDD Multi-Way Decision Diagrams

This novel data structure encodes all sets and functions encoded by the above mentioned styles of DD, providing a unified framework in which to exploit all their advantages, and is the subject of Section 3.1.

2.2 C++ template metaprogramming and library interface improvement

Extant libraries that implement the above-mentioned kinds of decision diagram are often somewhat cumbersome to use in model-checking code, as compared with what one might hope for, given that usually one merely needs to efficiently implement certain tuple-set-based operations.

Of special concern is the fact that, for the sake of efficiency, tuple-set operations must sometimes be performed where different arguments are different kinds of DD. Until now, these situations have been handled individually by manually implementing a new function to correctly calculate the desired function given the specific kinds of DDs used as input and desired as output. Thus, these libraries must be incrementally extended by their users when new combinations of parameter types are necessary. This situation results in a great may manually coded functions, each of which must be tediously debugged, etc., as well as in libraries where the API is bloated with many functions with similar purposes. This situation motivates our novel invention of GDDs, which have variants with the advantages of any of the above kinds of DD.

I propose to remedy the particular problem of having too many manually coded functions, through the use of GDDs and parametric variants thereof to describe the types of DDs and implement all associated operations through template metaprogramming from a relatively small base of manually generated code.

I further propose to improve the API of GDD libraries through the use of C++ template metaprogramming in a way that allows code to be modeled more closely on the higher-level pseudocode on which it is based.

Procedural *metaprogramming*, or the generation of code at compile time by user code, has been used in procedural languages ever since *macro* was added to the Lisp programming language, for the purpose of generating efficient code for specialized 'mini-language' notations embedded within procedural language code [34](§6.38). Later procedural languages, such as PL/1 and C also adopted the use of macros for code generation, although those macro implementations were relatively text-oriented and could easily generate unintended multiple definitions for a given name. The relative messiness of macros prompted the invention of 'hygienic' macro-expansion for languages in the Lisp family [30](§6.39)[2](§6.40)[5](§6.41) , while Ada and Java allow only *generics*, which have considerably less flexibility than macros, but are relatively safe [29](§6.44)[23](§6.45). The C++ programming language uses templates and type-based specialization to obtain many more of the benefits of macros in a relatively safe way [47](§6.42). The Scala programming language was created with extended support for metaprogramming in the form of implicit parameters, manifests, language feature virtualization, syntactic flexibility, and other features [39](§6.21)[35](§6.22). Scala would be my preferred language for this research if not for concerns about efficiency, as Scala implementations compile to Java bytecode. It has however been shown that C++ template specialization can be used to perform almost arbitrarily complex computation at compile time [44](§6.47). Thus C++ is in principle capable of supporting the kind of notational improvements I would like to implement, although with less syntactic convenience than what is possible with Lisp or Scala. The recent availability of implementations of C++11 further simplifies the task of metaprogramming in C++ [48](§6.43)[45](§6.46). Hence, I believe a C++11 GDD library implementation, using template metaprogramming, would fit well the needs of our model-checking research.

2.3 Saturation

The *Saturation* heuristic applies to fixed-point iteration problems where the solution is closed over many (preferably simple) operators. These problems may be solved by repetitively applying the operators as steps in any order, where applying any step does not diminish the progress due to later application of another step. That is, some steps applied in a certain order will cause the problem to be solved, and extra 'wrong' steps that occur in the process do not prevent solution.

The Saturation heuristic is controlled by a given ordering of the possible steps, usually corresponding to the approximate cost of a step. The heuristic simply applies inexpensive steps until they yield no change in the problem state, after which a more expensive step is tried. That is, inexpensive steps have priority over expensive steps. So, a step is applied only if application of all less expensive steps yields no change in the problem state. Many useful refinements to Saturation in the symbolic context are described in [10](§6.1).

Saturation has been found to be very effective for exploration of (certain kinds of) finite state spaces, when the state spaces are encoded symbolically (using DDs). Here, the problem state is encoded symbolically as a set of discovered states, and the steps are applications of state transition relations, also encoded symbolically. The SmArT tool is known for its use of Saturation to rapidly explore Petri Net state spaces with over 10^{20} states within seconds [11](§6.51).

Saturation has also been found effective for other model checking-related tasks, such as:

1. Calculating distance in large transition graphs [12](§6.50)
2. Strongly Connected Components via Transitive Closure of transition relations in large transition systems [55](§6.53)
3. Bisimulation with many (over 10^9) equivalence classes [37](§6.2)[38](§6.3)

2.4 Fully-symbolic algorithms for bisimulation and lumping

The author has previously explored application of the saturation heuristic to symbolic bisimulation, with some good results [37](§6.2)[38](§6.3). Significant progress in application of saturation to symbolic lumping requires using the advantages of Fully-Identity-Reduced MDDs in combination with Edge-Valued MDDs to model probabilistic transition relations, and partially motivates the current proposal. These efforts are briefly described below.

2.4.1 Bisimulation with deterministic transitions

A *bisimulation*[40] relation relates extensionally equivalent states in a labeled transition system. Exact extensional equivalence between states in given by the maximum bisimulation relation. The maximum bisimulation relation[33], $\sim \subseteq S \times S$ between sets of states S of a transition system with transition relations E (where $\forall(\xrightarrow{\alpha}) \in E : (\xrightarrow{\alpha}) \subseteq S \times S$), is defined as the largest equivalence relation B on S satisfying:

$$\forall(\xrightarrow{\alpha}) \in E : \forall\langle p, q\rangle \in B : \forall p' \in S : p \xrightarrow{\alpha} p' \Rightarrow \exists q' \in S : q \xrightarrow{\alpha} q' \wedge \langle p', q'\rangle \in B$$

In late 2008, I observed that when all transition relations were deterministic, as with individual transitions in Petri Nets, we have: $\forall(\xrightarrow{\alpha}) \in E : \forall\langle p, q\rangle \in \approx : \langle(\xrightarrow{\alpha})^{-1}(p), (\xrightarrow{\alpha})^{-1}(q)\rangle \in \approx$, thus, $\approx$ is closed over $(\xrightarrow{\alpha})^{-1} \times (\xrightarrow{\alpha})^{-1}$ for all $(\xrightarrow{\alpha}) \in E$.

I also showed that $\approx$ could be calculated by initializing a variable $\overline{B}$ to all pairs in $S \times S$ with different enablements or colorings, then applying the transitive closure of $(\xrightarrow{\alpha})^{-1} \times (\xrightarrow{\alpha})^{-1}$ to $\overline{B}$, iterating over all $(\xrightarrow{\alpha}) \in E$ according to the saturation heuristic.

The resulting 2011 paper [37](§6.2), showed good results for using the saturation heuristic to implement the transitive closure in the above strategy, resulting in the fastest known bisimulation algorithm when there are many resulting equivalence classes and the transition relations are deterministic.

2.4.2 Weak bisimulation

Weak bisimulation allows the possibility of 'invisible' transitions, which may occur without removing an input symbol. The largest weak bisimulation may be calculated using ordinary bisimulation algorithms, provided that the transition relations are preprocessed by appending the transitive closure of all invisible transition relations to each visible transition relation.

This preprocessing typically results in a transition system with nondeterministic transition relations. Nondeterministic transition relations may also arise with Petri Nets if multiple Petri Net transitions are given the same label in the transition system. Thus it is quite desirable to be able to efficiently calculate the largest bisimulation relation ($\sim \subseteq S \times S$) when some transitions are nondeterministic.

Closer analysis in 2009 showed that the definition of bisimulation can be reformulated as:

$$(\sim) \text{ is the largest relation } B \subseteq S \times S \text{ on } S \text{ satisfying:}$$
$$\forall (\xrightarrow{\alpha}) \in E : B = B \setminus \{\langle p, q\rangle | (\exists p' : (p \xrightarrow{\alpha} p' \wedge \neg \exists q' : q \xrightarrow{\alpha} q' \wedge p' B q')) \vee (\exists q' : (q \xrightarrow{\alpha} q' \wedge \neg \exists p' : p \xrightarrow{\alpha} p' \wedge q' B p'))\}$$

and that this formula was actually compatible with Saturation based implementation. My 2013 paper [38](§6.3), showed good results for using the resulting saturation-based algorithm on a variety of bisimulation problems. Important weaknesses were also identified, such as inefficiency in cases where some transition relations have very large support, as is often the case with weak bisimulation.

2.4.3 Lumping

Lumping is analogous to bisimulation, except that it calculates extensional equivalence between states of Markov systems rather than states of labeled transition systems. Lumping uses probabilistic transition matrices Q of type $S \times S \to \mathbb{R}$, instead of relations of type $S \times S \to \mathbb{B}$. The lumping equivalence relation may be defined as the largest equivalence relation B satisfying:

$$\forall Q \in E : \forall \langle p, q\rangle \in B : \forall r \in S : \sum_{p' | p' B r} Q(p, p') = \sum_{q' | q' B r} Q(q, q')$$

There is some similarity between symbolic lumping algorithms and symbolic bisimulation algorithms [54](§6.54)[16](§6.49). We expect that the enhanced TeDDy library will provide the necessary flexibility for attempting to solve the symbolic lumping problem, through adaption of our symbolic bisimulation algorithms.

2.5 Parallel DD algorithms

Model checking is frequently a computationally demanding task, so that space and speed improvements provided by symbolic methods do not suffice for all problems of interest. Parallel processing is another avenue by which researchers attempt to expand the range of solvable problems. A variety of approaches, for incorporating parallelism into symbolic model checking, have been explored. Most involve either of the following approaches:

1. Algorithm-level parallelism, where a (possibly sequential) DD library is used on each processor by an algorithm that employs some parallelism techniques to utilize multiple processors.

2. Library-level parallelism, where a (possibly sequential) algorithm invokes a parallelized DD library so that requested DD operations will be executed by the library code using multiple processors.

The RooMY[31](§6.9)[32](§6.10) system illustrates an interesting hybrid of both techniques. Kunkle et al implemented the RooMY system, which effectively utilizes the disks of multiple processors to carry out very large computations. They appear to use some parallelism-like techniques for organizing computation in a way that makes it less sensitive to the latency of lengthy disk accesses. By coherently grouping disk accesses, and using large RAM buffers, the efficiency of disk access is greatly improved, especially when compared to the use of disk as virtual memory for RAM-oriented algorithms. Kunkle et al then implemented a BDD library on top of RooMY, and subsequently solved some problems that could not previously be solved with RAM-based DD library implementations.

This is typical of the existing results for model checking using parallel DD libraries. In most cases, parallel implementations of a parallel algorithm obtain modest speedup compared with sequential implementations of the same algorithm, while comparison with carefully optimized sequential algorithms shows no significant speedup. Also, in many cases, a parallel algorithm is able utilize the RAM (or disks) of multiple machines to solve problems not solvable on a single processor.

There are roughly two strategies used to implement library level parallelism in parallel DD libraries, as described in the following two sub-subsections.

2.5.1 Distribution by level

Even in reduced decision diagrams, typically, the vast majority of edges lead from a node at one level to a node at the next lower level. As algorithms on DDs typically involve traversing the DD graphs through their edges, there is considerable temporal locality associated with connected nodes. In a distributed-memory processor network, distribution by level attempts to exploit this property by locating connected nodes on 'nearby' processors, meaning either the same processor, or processors sharing a physical direct communication link or a shared memory. [46](§6.15) uses this technique. In some cases, this associates a separate level of DD nodes with each processor. Although the matching of locality between the DD nodes with the locality of communication in the processor network produces a desirable limit on communication costs, the hoped-for speedups tend to not materialize. The algorithms for DD operations are usually organized as depth-first or breadth-first traversals of the DD graphs. In the depth-first case, the algorithm is intrinsically serial, so that no speedup is gained from using multiple processors. In the breadth-first case, parallel traversal activities may be spawned for each child of a given node, however this distribution scheme tends to place all of those child nodes in the same processor, again limiting opportunities for parallelism.

2.5.2 Distribution by function of state

These schemes place DD nodes among processors according to some function of their contents. In the case of library-level parallelism the function may be a hash involving the addresses of the node's children, so that the quasi-random node distribution maximizes the chances for parallel operations on any given collection of nodes. As most operations involve many nodes, this scheme is likely to produce much demand on a distributed communication network. In algorithm-level parallelism, different parts of a (partitioned) set may be stored as separate DDs and each DD placed by an algorithm-aware scheduler. This scheme may help to reduce inter-processor communication, but may encode a set less compactly due to the partitioning into multiple DDs.

2.6 Parallel symbolic state space exploration

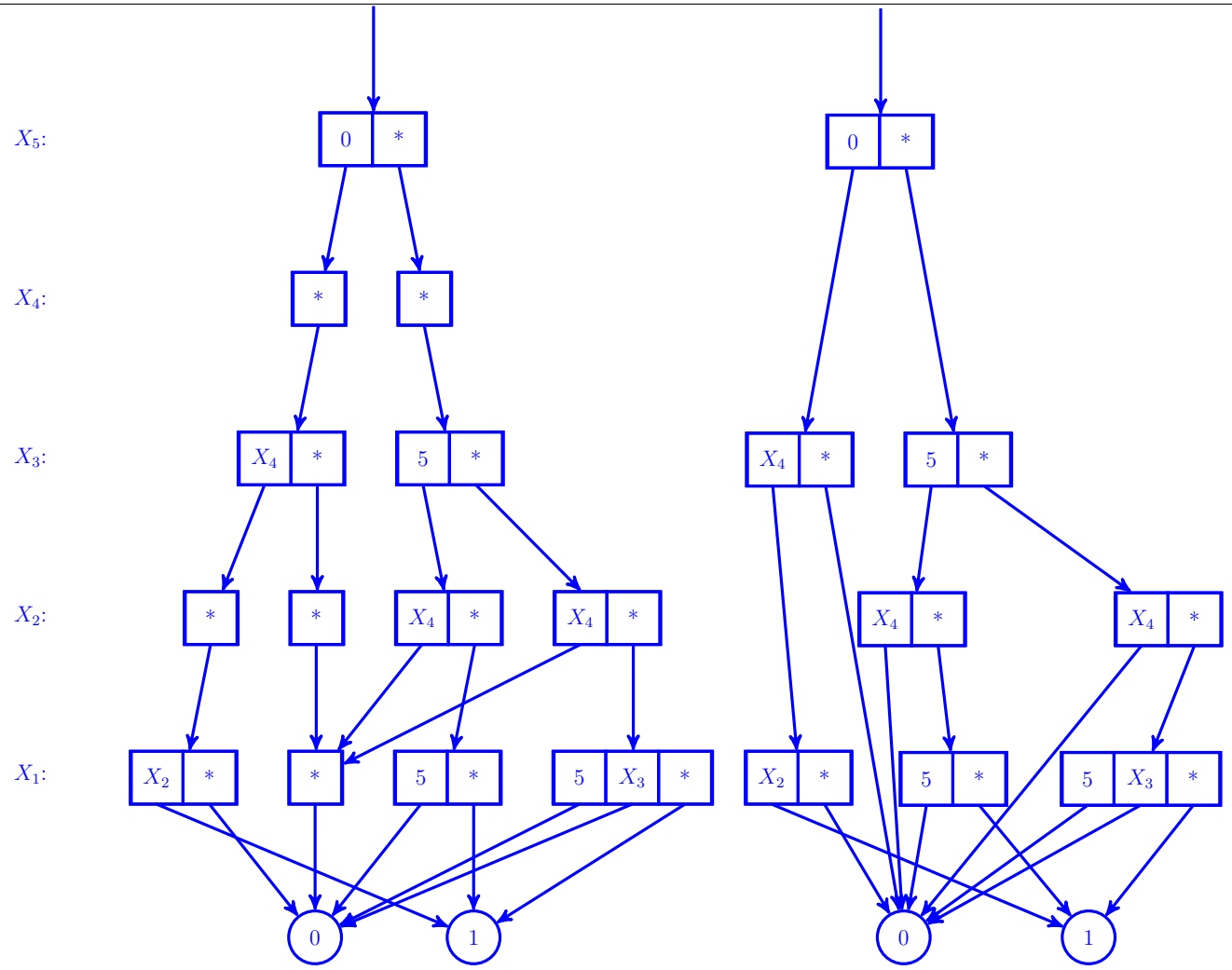
Saturation provides the fastest exploration of finite states spaces currently available on sequential machines. Because Saturation is so efficient in an often lazy recursive implementation, parallelization of Saturation has met with limited success. Ideas used to parallelize saturation have resulted in improvements to sequential saturation, nullifying potential parallel speed-ups [17](§6.13)[13](§6.14)[19](§6.17)[21](§6.18)[18](§6.19). So far, parallel saturation has provided access to more memory than what is available to individual computers, enabling solution of larger problems, but not providing hoped-for scalable parallel speed-up [**?**](§**??**)[**?**](§**??**)[13](§6.14)[19](§6.17)[21](§6.18)[18](§6.19).

Efforts by Grumberg et al, at Technion have obtained some useful speed-up using an approach similar to distribution by function of state described above [28](§6.4)[26](§6.5)[3](§6.6)[25](§6.7)[22](§6.8).

2.7 Space overhead of decision diagrams

Although the use of DDs to encode sets often saves space and time by many orders of magnitude, compared to an element list encoding, it is a heuristic compression technique, and space saving is not guaranteed. In the worst case, only minimal sharing occurs, so that a DD encoding of a set occupies more space than the corresponding element list encoding. This is especially true in the case of very small tuple-sets of large tuples.

As a simple example, consider the case of encoding the set $\{\langle 1,0,1,1,0,1\rangle\}$, as a BDD, and as an element list. The set has only one element, so the element list has a single node, of unspecified complexity, although 6 bits should suffice to encode the element in this case. The (un-reduced) BDD encoding, however, will have 6 levels (hence 6+ nodes), each of which must store 2 pointers. Thus we can see that un-reduced DDs have a space overhead proportional to the number variables in each tuple, for encoding tuple-sets. This overhead is not onerous if one is using DDs to encode sets with very many element tuples each of reasonable size. This overhead is excessive when encoding small sets of large tuples, and may account for the general lack of interest in symbolic encodings outside the model checking and logic programming community.



(a.) Quasi-reduced GDD encoding (b.) Fully-reduced GDD encoding

Fig. 6: $f(X) : \mathbb{N}^5 \to \mathbb{B} = (X_5 = 0 \wedge (X_4 = X_3 \wedge X_2 = X_1)) \vee (X_5 \neq 0 \wedge (X_4 \neq X_2 \wedge X_3 \neq X_1 \wedge X_1 \neq 5))$

3 Proposed contributions

Here I give the details of the novelty of my contribution.

3.1 (Generalized Decision) Diagrams

An (un-reduced) non-normalized *(Generalized Decision) Diagram* (GDD) A is a finite directed edge-labeled tree of constant depth (K), encoding a function f_A from members $\langle x_K, x_{K-1}, \ldots, x_1 \rangle$ of the domain of natural tuples of length K ($\mathbb{N}^K$) to some finite range R, where each leaf node is a member of R, and each edge label, from a node at distance k from a leaf, is either: (1) a (Natural) constant, (2) a variable name from the set of variables $\{`x'_K, \ldots, `x'_1\} \setminus \{`x'_k, \ldots, `x'_1\}$, or (3) '*', and each edge from a given node is uniquely labeled, and each node has one edge labeled with '*'. An (un-reduced) *normalized* GDD is a GDD that also satisfies the rules listed below in Section A.3.

A *quasi-reduced GDD* (QGDD) is a directed acyclic graph that represents an un-reduced GDD, and has the additional property that there are no redundant nodes, that is, identical nodes have been collapsed

to a common representation, so the resulting QGDD data structure contains no identical nodes. Fig. 6 shows sample (quasi-reduced) GDD encodings.

Notation: $labels(A)$ is the set of labels of edges from an node A, excluding '*' . For a non-leaf GDD (or QGDD) node A, we write $A\overrightarrow{p}$ to mean $p \in labels(A)$, and we write $A[p]$ for the node reached by the edge labeled p. Thus two nodes A and B are identical (written $A = B$) iff $labels(A) = labels(B) \wedge A[\text{'*'}] = B[\text{'*'}] \wedge \forall p \in labels(A) : A[p] = B[p]$. We write $level(A) = k$ iff the distance from A to a leaf node is k. We also write A to mean the entire tree at or below a GDD node A (or for all nodes reachable from an QGDD node A). We also write x_k for member k of a tuple x. GDD_k^K is the class of GDD nodes B where $level(B) = k$, encoding functions of K-tuples. With respect to a given level $k \leq K$, a K-tuple $x \in \mathbb{N}^K = \langle x_K, \ldots x_1 \rangle$ comprises a prefix $x_{\uparrow k} = \langle x_K, \ldots x_{k+1} \rangle \in \mathbb{N}^{K-k}$ and a suffix $x_{\downarrow k} = \langle x_k, \ldots x_1 \rangle \in \mathbb{N}^k$ and we say $x = x_{\uparrow k} x_{\downarrow k}$. We also impose a total ordering, $\succ$, on edge labels excluding '*', so that $\forall i, j \in \mathbb{N} : (i \succ j \text{ iff } i > j) \wedge (\text{'}x\text{'}_i \succ \text{'}x\text{'}_j \text{ iff } i > j) \wedge (i \succ \text{'}x\text{'}_j)$.

3.1.1 Decoding of GDDs

The meaning of a GDD A corresponds to the set of paths, from the root of A to leaves, whose edge labels match corresponding elements of the argument x of the function f_A encoded by A.

I first define the helper function *match* to indicate which edges match a given argument. I write $match(A, p, x)$ to mean that the edge labeled p from node A matches tuple x. Informally, $match(A, p, x)$ holds when the member, $x_{level(A)}$ of x has the value p (when $p \in \mathbb{N}$), or the value of the variable, within x, named by p, when p is a variable name.

I first define *eval* on labels other than '*' to simplify the definition of *match*. *eval* of an edge label p at level k also requires a tuple prefix (indicated as a subscript) used for evaluation of symbolic labels.

$$eval_{x\uparrow k}(\text{'}x\text{'}_i) = x_i \tag{1}$$

$$eval_{x\uparrow k}(c) = c \tag{2}$$

match has type: $\forall k, K | k \leq K : GDD_k^K \times (\mathbb{N} \cup \{\text{'}x\text{'}_K, \ldots, \text{'}x\text{'}_1\} \setminus \{\text{'}x\text{'}_k, \ldots, \text{'}x\text{'}_1\}) \times \mathbb{N}^K \rightarrow \{true, false\}$, and is defined as follows:

$$match(A, p, x) = A\overrightarrow{p} \wedge (eval_{x\uparrow k}(p) = x_k) \tag{3}$$

Thus, *match* requires A to have an outgoing edge label p that evaluates to x_k, in the context $x{\uparrow}k$.

The function f_A, of type $\mathbb{N}^K \rightarrow R \cup \{AmbiguityError\}$, encoded by a GDD A, where $level(A) = k$, is defined recursively as follows:

when $k = 0$, (A is a leaf node)

$$f_A(x) = A \tag{4}$$

when $k > 0 \wedge match(A, p, x)$ for exactly one edge label p, (A has one edge label matching x_k)

$$f_A(x) = f_{A[p]}(x) \tag{5}$$

when $k > 0 \wedge match(A, p, x) \wedge match(A, q, x) \wedge p \neq q$ for edge labels p, q, (multiple edges match x_k)

$$f_A(x) = AmbiguityError \tag{6}$$

otherwise (when $match(A, p, x)$ for no edge label p), (A has no edges matching x_k)

$$f_A(x) = f_{A[\text{'*'}]}(x) \tag{7}$$

Thus, except when *AmbiguityError* is encountered, the value of f at a non-leaf node is the value of f at one of its children, so that the value depends on the path, selected by members of the tuple, through the tree from the root to a leaf. Note that the choice of this path depends only on conditions of the forms: $x_k = c$ (for some constant c present in the GDD) and $x_k = x_i$ (for an index $i > k$), and their complements. Thus, for any given GDD, any two tuples which induce the same choices for such conditions will result in the in the choice of the same path.

The *AmbiguityError* of Equation 6 occurrs when $eval_{x\uparrow k}(p) = eval_{x\uparrow k}(q)$ for two different non-'*' edge labels, p and q, from A. This can occur in two ways:

(1) $p = c$, for some constant $c \in \mathbb{N}$ and $x_k = c$, while $q = \text{`}x\text{'}_j$ for some $K \geq j > k$ and $x_j = c$, hence $x_k = x_j$. Thus a constant-labeled edge and a variable-labeled edge might both match the same argument. This ambiguity can be avoided by requiring $c \neq x_j$ to hold of the argument x of f_A

(2) $p = \text{`}x\text{'}_i$ for some $K \geq i > k$ and $x_k = x_i$, while $q = \text{`}x\text{'}_j$ for some $K \geq j > k$ and $x_k = x_j$, hence $x_k = x_i = x_j$. Thus two different variable-labeled edges might also both match the same argument. Similarly to the first case, this ambiguity can be avoided by requiring $x_i \neq x_j$ to hold of the argument x of f_A

The case where p and q are both constant labels never produces ambiguity because of the requirement that all edges from a given node have unique labels.

To ensure unambiguous definition of f, we associate, with each node A, a finite set of inequality constraints C_A, which will limit the domain of f_A. The constraints in C_A, where $level(A) = k$ are of 2 forms: (1) $c \neq x_j$ for some constant integer c and some $j > k$, and (2) $x_i \neq x_j$ for some $i > k$ and some $j > k$. We consider $p \neq q$ and $q \neq p$ to be the same constraint, for any labels p and q,

C_A is populated with exactly those constraints required by the following rules, which suffice to ensure the function f_A, encoded by GDD A, is well defined:

1. When $level(A) = 0$, $C_A = \emptyset$.

   No constraints are required to ensure $f_A \neq AmbiguityError$ in this case.

2. For each node A, index j and label p where $A\overrightarrow{p} \wedge A\overrightarrow{\text{`}x\text{'}_j} \wedge (p \neq \text{`}x\text{'}_j)$, we have $(p \neq x_j) \in C_A$.

   Two edge labels must not become equivalent within the context of a given input. Thus, we constrain the input x so that Equation 6 will not immediately apply when applying f_A.

3. For any node A with $level(A) = k$, and any label p and index $j > k$, we have $A\overrightarrow{\text{`}x\text{'}_j} \wedge (x_k \neq p) \in C_{A[\text{`}x\text{'}_j]} \Rightarrow (x_j \neq p) \in C_A$.

4. For any node A with $level(A) = k$, and any labels p, q, r, we have $(A\overrightarrow{r} \vee r = \text{`*'}) \wedge (p \neq q) \in C_{A[r]} \wedge p \neq \text{`}x\text{'}_k \neq q \Rightarrow (p \neq q) \in C_A$.

5. For any node A with $level(A) = k$, and any label p, we have $(x_k \neq p) \in C_{A[\text{`*'}]} \Rightarrow A\overrightarrow{p}$.

Note that, for GDDs B with $level(B) = k$, these rules introduce constraints into C_B that involve only variables x_i with $i > k$, while such constraints are absorbed or transformed at level i, eliminating any constraints involving x_i from $C_{B'}$ (where $level(B') = i$), so that, when $level(A) = K$, $C_A = \emptyset$ always.

Thus, f_A, where A is a 'top level' GDD with $level(A) = K$, is always defined over the entire K-dimensional space of natural K-tuples, $\mathbb{N}^K$, while f_B, where B is a 'lower level' GDD with $level(B) = k < K$, is only defined for those members x of $\mathbb{N}^K$ which satisfy the constraints in C_B.

This suffices, because the recursive decoding process never uses f_B if the argument does not satisfy C_B. This will be shown after the remaining rule 6 is given.

Since C_B involves only variables x_i with $i > k$, where $level(B) = k$, satisfaction of C_B by a tuple x, is actually a property of the prefix $x_{\uparrow k}$.

I will therefore write $sat(B)$ to mean the set of prefixes that satisfy the inequality constraints C_B. Thus, $x_{\uparrow level(A)} \in sat(A)$ implies x satisfies the inequality constraints C_A. I will also write $x \in sat(A)$ to mean the same thing.

The above rules still do not eliminate the possibility of an *inconsistent* condition, where an edge e labeled p from a node A at level k leads to a node with a constraint such as $x_k \neq p$, so that no tuple satisfying the constraint has a value for x_k which permits e to be followed. The following rule suffices for avoiding such inconsistencies:

6. For each label $p \in labels(A)$ (so $p \neq$ '*'), the entire tree under $A[p]$ has no edges labeled 'x'$_k$, where $level(A) = k$.

Note that this makes rule 3 irrelevant, as constraints on x_k can arise only from a tree where it occurs. Rule 3 would be relevant if I had chosen the following alternative rules instead of rule 6:

6'a. For each constant c where $A\overrightarrow{c} \wedge level(A) = k$, the entire tree under $A[c]$ has no edges labeled 'x'$_k$.

6'b. For any node A with $level(A) = k$, and each index $j > k$ where $A\overrightarrow{\text{'}x\text{'}_j}$, the entire tree under $A[$'x'$_j]$ has no edges labeled 'x'$_j$

I chose[1] rule 6 over rules 6' to simplify the canonicity proof in Section A.3.

An *unambiguous GDD* is a GDD that follows rules 1-6 above. Hereafter, all GDDs are unambiguous except where clearly noted.

The following mathematical details (11 pages) are relegated to the appendix:

In Section A.1, I show that for any unambiguous GDD A, with child B, the recursive decoding process for f_A never uses f_B if the argument x does not satisfy C_B. In Section A.2, I define notions necessary for showing canonicity, while in Section A.3, I define (quasi-reduced) *normalized* GDDs, and show they are canonical. In Section A.4, I define the domain (*BundleUnions*) over which GDD-encoded functions apply, and show in Section A.5 that tuple-sets represented by GDD-encoded characteristic functions are closed over union, intersection, complement, and cartesian products.

Hereafter, all GDDs are quasi-reduced normalized GDDs unless explicitly noted otherwise.

3.1.2 Adequacy of GDDs

Choosing the range $R = \{true, false\}$ leads to encoding of boolean functions of tuples, which may be taken as characteristic functions of tuple sets. Thus tuple sets and tuple relations may be encoded by GDDs.

GDDs encode all sets encoded by extensible QMDDs. This is fairly obvious, as QMDDs are similar to a special case of QGDDs, where no edge labels are variable names. In this case, ambiguity rules 1-6 in Section 3.1.1, (hence canonicity rule 1) are automatically satisfied with $C_A = \emptyset$ for all nodes A. Canonicity rule 2 remains unchanged.

GDDs also encode all sets encoded by FI MDDs, although this case is more complex. In lieu of a proof, note that, as with GDDs, each path through a FI-reduced MDD obviously corresponds to a Bundle, so that all characteristic functions of sets encoded by FI-reduced MDDs are Bundle-wise constant.

GDDs are closed over complement, union, intersection, cartesian product, and product with the full set, due to the fact that they encode exactly all Bundle-wise constant functions.

[1] Analysis of the consequences of these rules seems to indicate that the alternate rule 6' has significant advantages for sharing of substructures. However, I believe these advantages are available to both alternatives, after an additional layer of encoding is used in the implementation of the physical data structures.

3.1.3 Reductions with GDDs

All the reductions mentioned in Section 2.1, and more, may be applied on a level-by-level basis to GDDs in a uniform manner, so that distinct algorithms to handle each reduction are no longer necessary. In each case, an edge that skips a particular level is considered to be an abbreviated form for a very specific form of node, as follows:

1. Quasi-reduced is an essentially un-reduced form of DD not allowing skipped levels.

2. Fully-reduced level of a GDD. When level k is Fully-reduced, an edge from node A which skips level k and leads to a node C at level $j < k$ is equivalent to the same edge from node A leading instead to a node B at level k, where B has a single edge leading to C, labeled '$*$'.

3. Identity-reduced level of a GDD. When level k is Identity-reduced, an edge from node A which skips level k and leads to a node C at level $j < k$ is equivalent to the same edge from node A leading instead to a node B at level k, where B has 2 edges, one labeled 'X_{k+1}' leading to C, and the other labeled '$*$' leading to an encoding of the empty set.

4. Constant-reduced level of a GDD. When level k is Constant(c)-reduced, an edge from node A which skips level k and leads to a node C at level $j < k$ is equivalent to the same edge from node A leading instead to a node B at level k, where B has 2 edges, one labeled 'c' leading to C, and the other labeled '$*$' leading to an encoding of the empty set.

For purposes of maintaining canonicity, obviously the 'equivalent' node B must never exist, as that would create multiple encodings for a given function. A Fully-reduced level of a GDD, for example, must never have a node with only a single edge labeled '$*$', since a path through such a node should simply skip that level.

The use of the 'empty set', meaning a GDD encoding of a function that always returns 0, in items 3 and 4 points out the asymmetric nature of null-pointer elimination. The null pointer customarily is used to encode a function that always returns 0, but there is no such simple customary encoding for a function that always returns a constant other than 0. Hence, the complement operation, which nominally can be implemented by exchange of 'true' and 'false' leaves, must, in practice include special code to process cases involving encodings of the empty set or of the 'full' set. I expect to remedy this asymmetry in my GDD implementation, either by not using null pointers, or by providing a simple encoding for all functions that always return a constant.

3.1.4 Edge-valued GDDs

It is somewhat obvious that an edge-valued variation of GDDs (EVGDDs?), having the benefits of both EVMDDs and GDDs, is possible, and that a library for manipulating such data structures could be conveniently implemented as an increment in the current proposal.

3.1.5 Algorithms for GDDs

Algorithms for GDD manipulation are generally expected to resemble those for MDD manipulation except for the following characteristic. Meanings of subgraphs of GDDs obviously have more context dependence than meanings of subgraphs of MDDs. This dependence causes non-trivial operations (such as set union) to have multiple values when they are implemented in a context-independent manner. Each result value is valid in certain contexts associated with possible sets of constraints on 'higher' variables not accessed by the operation. I anticipate that all of the multiple values of an operation will be cached for later use.

Due to time constraints I must defer the writing of efficient algorithms to a point after this proposal. However, from 2014-July-17 through 2014-Aug-7, I inserted the following algorithm for union, since I have been working on it since that start of July, according to the proposed schedule in Plan D (Section 5.1.2).

The *GDDUnion* algorithm calculates the set union of a number of sets, all represented as GDDs encoding their characteristic functions. The result is multivalued, where the value might depend on the prefix, so the result is a (possibly GDD-encoded) mapping of the prefix onto the characteristic function representing the resulting set that applies to the particular prefix. For any GDD node $A \in GDD_k^K$, $\mathbb{C}_A$ refers to the set of K-tuples d where the prefix, $d_{\uparrow k}$ (hence also the tuple d) satisfies the constraint set C_A defined in Section 3.1.1, thus $\mathbb{C}_A$ is the set of tuples where the function f_A is defined.

memoized Function $GDDUnion(K, k, a_1, a_2, \ldots, a_n)$ (union of n arguments, for small n)
$: \mathbb{N} \times \{1, \ldots, K\} \times GDD_k^K \times GDD_k^K \to ((\mathbb{C}_{a_1} \cap \mathbb{C}_{a_2} \cap \ldots \cap \mathbb{C}_{a_n}) \to GDD_k^K)$ is:
let $D = \mathbb{C}_{a_1} \cap \mathbb{C}_{a_2} \cap \ldots \cap \mathbb{C}_{a_n}$ (domain of result function is a Bundle)

(several checks for 'easy' special cases that return without caching result) (Only the first two are necessary. The other special cases are optional optimizations, as those cases are correctly handled by the general case after the endif.)

if $(n = 0)$ then: (degenerate case)

Return $GDDUnion(K, k) =$ encoding of empty set

else $(n > 0)$

if $(K = 0)$ then: (leaf case where $a_1, a_2, \ldots, a_n \in \mathbb{B}$)

Return $Union(K, k, a_1, a_2, \ldots, a_n) = \bigvee_{i \in \{1 \ldots n\}} a_i)$

else $(K > 0)$

if $(n = 1)$ then: (simplest non-degenerate non-leaf case)

Return $GDDUnion(K, k, a_1) = a_1$

else $(n > 1)$

if $(\exists i \in \{1 \ldots n\} : \forall x \in D : f_{a_i}(x) = 1)$ then: (an argument encodes full set)

Return $Union(K, k, a_1, a_2, \ldots, a_n) = a_i)$ (encoded full set)

else (no full sets)

if $(\exists i \in \{1 \ldots n\} : \forall x \in D : f_{a_i}(x) = 0)$ then: (an argument encodes empty set)

Return $Union(K, k, a_1, a_2, \ldots, a_n) = Union(K, k, a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$ (remove redundant argument)

else (no empty sets)

(sorting the arguments by some arbitrary unique key may be useful here, and for cache simplification)

if $\exists i, j \in \{1 \dots n\}, j \neq i : a_i = a_j$ then: (two arguments are identical)

Return $Union(K, k, a_1, a_2, \dots, a_n) = Union(K, k, a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n)$ (remove redundant argument)

else

endif (Not one of the 'easy' special cases)

(proceed to handle the general case)

(first, partition the domain $D = \mathbb{C}_{a_1} \cap \mathbb{C}_{a_2} \cap \ldots \cap \mathbb{C}_{a_n}$ as necessary according to prefix,

depending on the labels of the outgoing edges from the nodes $a_1, a_2, \ldots, a_n$)

(To partition D, identify possible equivalence constraints for each variable)

Let $L = \bigcup_{i \in \{1 \ldots n\}} labels(a_i)$ (all the labels of the outgoing edges from $a_1, a_2, \ldots, a_n$)

For each $k' \in K \ldots k+1$, where 'x'$_{k'} \in L$:

Let $L[k'] = \{l : L | \neg \exists_{i \in \{1 \ldots n\}} \{\text{'}x\text{'}_{k'}\} \cup \{l\} \subseteq labels(a_i)\}$

($L[k']$ has all labels which might sometimes have the same value as 'x'$_{k'}$)

(now partition D)

Assign $P \leftarrow \{D\}$ (the partition of D with a single equivalence class D.)

For each $k' \in K \ldots k+1$, where 'x'$_{k'} \in L$: And $p \in P$:

(Enumerate possible label values that may equal 'x'$_{k'}$ given p)

Let $V = \{v_1, v_2, \ldots, v_m\} = \{l \in L[k'] | \exists X \in p : eval_X(\text{'}x\text{'}_{k'}) = eval_X(l)\}$

(Split p according to possible values of $x_{k'}$)

Replace p in P with $\{X \in p | eval_X(\text{'}x\text{'}_{k'}) = eval_X(v_1)\}$, $\{X \in p | eval_X(\text{'}x\text{'}_{k'}) = eval_X(v_2)\}$ $\ldots$ $\{X \in p | eval_X(\text{'}x\text{'}_{k'}) = eval_X(v_m)\}$, $\{X \in p | eval_X(\text{'}x\text{'}_{k'}) \neq eval_X(v_1) \wedge eval_X(\text{'}x\text{'}_{k'}) \neq eval_X(v_2), \ldots, \wedge eval_X(\text{'}x\text{'}_{k'}) \neq eval_X(v_m)\}$

(P is now a partition of D according to the relevant edge labels)

For each $p \in P$: construct the pseudo-GDD F_p (where each $F_p[]$ is a function mapping p to a (GDD^K_{k-1}) potential child instead of $F_p[]$ being an actual child.) (Yes, I am abusing my own notations.)

(recursively) compute $F_p[\text{'*'}] = GDDUnion(K, k-1, a_1[\text{'*'}], a_2[\text{'*'}], \ldots, a_n[\text{'*'}])$

Let $labels(F_p) = \{x_k \in L | \neg \exists x'_k \in L : (x'_k \succ x_k \wedge \forall x \in p : eval_x(x_k) = eval_x(x'_k))\}$

For each $x_k \in labels(F_p)$:

(recursively) compute $F_p[x_k] = GDDUnion(K, k-1, a_1[x''_k], a_2[x''_k], \ldots, a_n[x''_k])$,

where $x''_k \in labels(a_i) \wedge \forall x \in p : eval_x(x''_k) = eval_x(x_k)$ (There is at most one such x''_k)

otherwise $x''_k = \text{'*'}$ (use label '*' when above does not hold)

(For each label x_k where $F_p[x_k]$ is computed, $F_p[x_k]$ has type: $\{y \in p | y_k = x_k\} \rightarrow GDD^K_{k-1}$, and, if none of the F has any dependence on 'higher' variables, $F_p[x_k]$ would map all of $\{y \in p | y_k = x_k\}$ to the GDD node encoding the function for subdomain $\{y \in p | y_k = x_k\}$, while $F_p[\text{'*'}]$ would map the remainder of p to the GDD node encoding the function for the remainder of p.)

(Now restructure P by subdividing each $p \in P$ according to all the subdomains in all the $F_p[]$'s)

(By subdomain, I mean the set of domain values mapping to the same range element)

Assign $P' \leftarrow P$

For each $p \in P$: For each $x_k \in labels(F_p) \cup \{\text{`*'}\}$: For each sub-domain $d' \in$ the domain of $F_p[x_k]$: For each $p' \in P'$ where $p' \subseteq p$:

if $p' \cap d' \neq \emptyset \wedge p' \setminus d' \neq \emptyset$ then replace p' in P' with $p' \cap d'$, $p' \setminus d'$ (split p' using d')

(Each $p' \in P'$ will be used as a subdomain of the domain D of the output mapping)

(Finally construct the output mapping $f : D \to GDD_k^K$))

Assign $f \leftarrow \emptyset$ (initialize)

For each $p' \in P'$: (construct node used in $f(p')$)

Let $p \in P \wedge p' \subseteq p$ (There is exactly one such p)

GDD_k^K node $F'_{p'}$, where $labels(F'_{p'}) = labels(F_p)$

For each $x_k \in labels(F'_{p'}) \cup \{\text{`*'}\}$: (Choose child for label x_k)

Assign $F'_{p'}[x_k] \leftarrow F_p[x_k](y)$ (for any $y \in p'$ where $x_k = y_k$) (All such y give the same result)

Assign $f \leftarrow f \cup \{x \mapsto unique(F'_{p'}) : x \in p'\}$

Return $GDDUnion(K, k, a_1, a_2, \ldots, a_n) = f$ (and memoize result)

3.2 TeDDy interface improvement through C++ metaprogramming

In lieu of a formal description of the details of the proposed improvements, consider the following algorithm, as published in [38](§6.3):

“

MDD $Bisim$(level k, MDD $\overline{\mathcal{T}}_{\mathcal{E}}$, MDD $\mathcal{B}_{in}$) is — • *memoize this function*

local MDD $\mathcal{B}$, MDD $\mathcal{B}_{old}$, MDD $\mathcal{Z}$, MDD $\mathcal{Z}^R$, MDD $\mathcal{D}$, MDD $\mathcal{D}^R$

1 if $k = 0$ then return $\mathcal{B}_{in}$; — • *leaf case*
2 $\mathcal{B} \leftarrow \mathcal{B}_{in}$;
3 $\mathcal{B} \leftarrow BisimSaturate(k, \overline{\mathcal{T}}_{\mathcal{E}}, \mathcal{B})$; — • *saturate below this level*
4 repeat
5 $\mathcal{B}_{old} \leftarrow \mathcal{B}$;
6 for each $e \in \mathcal{E}$ where $Top(\ \overline{\mathcal{T}}_e\ ) = k$, loop — • *let* $p \xrightarrow{\overline{e}} p' \triangleq \langle p, p' \rangle \in \overline{\mathcal{T}}_e$
7 $\mathcal{Z} \leftarrow \{(p', q) | \exists q' : q \xrightarrow{\overline{e}} q' \wedge p' \mathcal{B} q'\}$; — • *in 1 symbolic step (composition)*
8 $\mathcal{Z}^R \leftarrow \{(p, q') | \exists p' : p \xrightarrow{\overline{e}} p' \wedge p' \mathcal{B} q'\}$; — • *also 1 step composition*
9 $\mathcal{D} \leftarrow \{(p, q) | \exists p' : (q \in \mathcal{S}_k \times \ldots \times \mathcal{S}_1) \wedge (p \xrightarrow{\overline{e}} p') \wedge \neg(p' \mathcal{Z} q)\}$; — • *also 1 step*
10 $\mathcal{D}^R \leftarrow \{(p, q) | \exists q' : (p \in \mathcal{S}_k \times \ldots \times \mathcal{S}_1) \wedge (q \xrightarrow{\overline{e}} q') \wedge \neg(p \mathcal{Z}^R q')\}$; — • *also 1 step*
11 $\mathcal{B} \leftarrow \mathcal{B} \setminus (\mathcal{D} \cup \mathcal{D}^R)$; — • *refine*
12 $\mathcal{B} \leftarrow BisimSaturate(k, \overline{\mathcal{T}}_{\mathcal{E}}, \mathcal{B})$; — • *re-saturate*
13 end loop;
14 until $\mathcal{B}_{old} = \mathcal{B}$;
15 return $\mathcal{B}$;

Fig. 2. Bisimulation algorithm using Saturation heuristic.

MDD $BisimSaturate$(level k, MDD $\overline{\mathcal{T}}_{\mathcal{E}}$, MDD $\mathcal{B}$) is

local MDD $\mathcal{B}'$, MDD $\mathcal{B}_*$, MDD $\mathcal{B}'_*$

1 $\mathcal{B}' \leftarrow$ new empty mutable MDD node;
2 for each index L in $\mathcal{B}$ loop — • *loop over children*
3 $\mathcal{B}_* \leftarrow \mathcal{B}_L$;
4 $\mathcal{B}'_* \leftarrow$ new empty mutable MDD node;
5 for each index R in $\mathcal{B}_*$ loop — • *loop over grandchildren*
6 $\mathcal{B}'_{*R} \leftarrow Bisim(k-1, \overline{\mathcal{T}}_{\mathcal{E}}, \mathcal{B}_{*R})$;
7 end loop;
8 $\mathcal{B}'_L \leftarrow unique(\mathcal{B}'_*)$;
9 end loop;
10 return $unique(\mathcal{B}')$;

Fig. 3. Helper function for Bisimulation algorithm.

”

My saturation-based bisimulation algorithm, exactly as printed in [38](§6.3).

The first C++ version of the code inside *Bisim* was (after removing the most embarrassing parts):

// Parameters: int L, MDDL::mdd hatS, MDDL::mdd * T, int NQ, MDDL::mdd Bin, MDDL::OperatorCode cache

```
// calculate T2 and NQ2, for lower levels
MDDL::mdd * T2 = T;
int NQ2 = NQ;
const int hatSL = hatS.level();
const int SXSL = hatSL+hatSL;
while( NQ2 && (T2[0].level()==SXSL)){ T2++; NQ2--; };
//convenience:
int L3 = L+L+L;
// utility stuff:
bool * oddlevels = levelsmod( levelsmod<bool>( NULL, 4*L, false, 1, 0), 4*L, true, 2, 1 ); oddlevels[0]=false;
bool * alllevels = levelsmod<bool>( NULL, 4*L, true, 1, 0); alllevels[0]=false;
// mod 3 utility stuff:
bool * levels1 = levelsmod( levelsmod<bool>( NULL, 3*L, false, 1, 0), 3*L, true, 3, 1 ); levels1[0]=false; // variable 3rd of 3
bool * levels2 = levelsmod( levelsmod<bool>( NULL, 3*L, false, 1, 0), 3*L, true, 3, 2 ); levels2[0]=false; // variable 2nd of 3
bool * levels3 = levelsmod( levelsmod<bool>( NULL, 3*L, false, 1, 0), 3*L, true, 3, 0 ); levels3[0]=false; // variable 1st of 3
bool * levels12 = levelsmod( levelsmod( levelsmod<bool>( NULL, 3*L, false, 1, 0), 3*L, true, 3, 1 ), 3*L, true, 3, 2 ); levels12[0]=false;
bool * levels23 = levelsmod( levelsmod( levelsmod<bool>( NULL, 3*L, false, 1, 0), 3*L, true, 3, 2 ), 3*L, true, 3, 0 ); levels23[0]=false;
bool * levels13 = levelsmod( levelsmod( levelsmod<bool>( NULL, 3*L, false, 1, 0), 3*L, true, 3, 1 ), 3*L, true, 3, 0 ); levels13[0]=false;
int * sig2Lop = levelsmod( levelsmod( levelsmod<int>( NULL, 3*L, (int)0x88888888, 1, 0), 3*L, (int)0xAAAAAAAA, 3, 0), 3*L, (int)0xCCCCCCCC, 3, 1); sig2Lop[0]=0x88888888; //variab
int * sig2Luse = levelsmod( levelsmod( levelsmod<int>( NULL, 3*L, 3, 1, 0), 3*L, 1, 3, 0), 3*L, 2, 3, 1); sig2Luse[0]=3; //variable1: 0; variable2: 1,0; variable3: 1;
int * sig2Rop = levelsmod( levelsmod( levelsmod<int>( NULL, 3*L, (int)0x88888888, 1, 0), 3*L, (int)0xAAAAAAAA, 3, 0), 3*L, (int)0xCCCCCCCC, 3, 2); sig2Rop[0]=0x88888888; //variab
int * sig2Ruse = levelsmod( levelsmod( levelsmod<int>( NULL, 3*L, 3, 1, 0), 3*L, 1, 3, 0), 3*L, 2, 3, 2); sig2Ruse[0]=3; //variable1: 0; variable2: 1; variable3: 1,0;
int * DeltaBbarop = levelsmod<int>( NULL, 3*L, (int)0xFFFFFFFE, 1, 0); DeltaBbarop[0]=0x4F444F44; //variable1: 3|1; variable2: 3|1; variable3: 3|1; variable 0: (3&!2)|(1&!0);
int * DeltaBbaruse = levelsmod( levelsmod( levelsmod<int>( NULL, 3*L, 15, 1, 0), 3*L, 11, 3, 0), 3*L, 14, 3, 2); DeltaBbaruse[0]=15; //variable1: 3,1,0; variable2: 3,2,1,0; varia
int * DeltaBbaralt2use = levelsmod( levelsmod( levelsmod<int>( NULL, 3*L, 15, 1, 0), 3*L, 15, 3, 0), 3*L, 15, 3, 1); DeltaBbaralt2use[0]=15; //variable2: 3,2,1,0 everywhere;

MDDL::mdd B = Bin;
MDDL::mdd Bold = B;

// first, saturate lower levels:
B = BiSat1Saturation( L, hatS, T2, NQ2, B, cache );

// then do fixed point calculation:
do {
    Bold = B;
    // operate at this level:
    // apply each transition relation at this level (T is sorted by descending Top level):
    for ( int a=0; (a<NQ) && (T[a].level() == SXSL); ++a ) {
        MDDL::mdd Ta = T[a];

        // sig1L = (B X hatS) & (T X2 hatS) // actually just (T X2 hatS)
        MDDL::mdd sig1L = InsertDontCaresQQ( L3, Ta, levels2, hatS, MDD_INSERTDC5_QQ );

        // sig1R = (B X hats) & (hats X T) // actually just (hats X T)
        MDDL::mdd sig1R = InsertDontCaresQQ( L3, Ta, levels3, hatS, MDD_INSERTDC3_QQ );

        // sig2L = (B o T) X2 hatS // actually just (B o T)
        // sig2Lpre = (B o T)pre = (* X B) & (T X *)
        MDDL::mdd sig2Lpre = GenericCompose4QQ( 0, 0, B, Ta, sig2Luse, sig2Lop, L3, MDD_GCOMPOSE43_QQ );
        MDDL::mdd sig2L = ProjectUnionQQ( sig2Lpre, levels2, MDD_PROJECTU5_QQ );
        MDDL::mdd sig2Lalt = InsertDontCaresQQ( L3, sig2L, levels2, hatS, MDD_INSERTDC5_QQ );

        // sig2R = hatS X (Binv o T) // actually just (Binv o T)
        // sig2Rpre = (Binv o T)pre = (* X B) & (T X2 *)
        MDDL::mdd sig2Rpre = GenericCompose4QQ( 0, 0, B, Ta, sig2Ruse, sig2Rop, L3, MDD_GCOMPOSE44_QQ );
        MDDL::mdd sig2R = ProjectUnionQQ( sig2Rpre, levels1, MDD_PROJECTU4_QQ );
        MDDL::mdd sig2Ralt = InsertDontCaresQQ( L3, sig2R, levels3, hatS, MDD_INSERTDC3_QQ );

        // DeltaBbarpre = sig1L\(hatS X sig2R) U sig1R\(sig2L X2 hatS)
        MDDL_ASSERT((hatS));
        MDDL::mdd DeltaBbarpre = GenericCompose4QQ( sig1L, sig2R, sig1R, sig2L, DeltaBbaruse, DeltaBbarop, L3, MDD_GCOMPOSE45_QQ );
        MDDL::mdd DeltaBbarpre1alt = MDDL::g_mddf.minus_qq(sig1R,sig2Lalt);
        MDDL::mdd DeltaBbarpre2alt = MDDL::g_mddf.minus_qq(sig1L,sig2Ralt);
        MDDL::mdd DeltaBbarprealt = MDDL::g_mddf.or_qq(DeltaBbarpre2alt,DeltaBbarpre1alt);

        MDDL::mdd DeltaBbar = ProjectUnionQQ( DeltaBbarpre, levels1, MDD_PROJECTU4_QQ );

        // put the results into B
        B = MDDL::g_mddf.minus_qq( B, DeltaBbar );

        // saturate those results:
        {if(B) B = BiSat1Saturation( L, hatS, T2, NQ2, B, cache ); };
        //MDDL_ASSERT((B));
    };
} while ( Bold != B );

// destruct utilities
CDEL(DeltaBbaruse);
CDEL(DeltaBbarop);
CDEL(sig2Ruse);
CDEL(sig2Rop);
CDEL(sig2Luse);
CDEL(sig2Lop);
CDEL(levels13);
CDEL(levels23);
CDEL(levels12);
CDEL(levels3); CDEL(levels2); CDEL(levels1);
CDEL(alllevels); CDEL(oddlevels);

MDDL::g_mddf.cache_add( cache, Bin, B );  // memoize
return B;
```

Note the amorphous block of code near the beginning, containing many calls to the levelsmod function. That code calculates the MDD equivalent of 'dope vectors', used in later operations to indicate choices associated with individual MDD levels of operands in later MDD operations. Those operations are mostly set and relational compositions of various kinds within the nested do-while and for loops. These calls are located above the loop, as the vectors are independent of the loop iteration in which they are used. Generation of this code is very error-prone and tedious. I made a number of templates to simplify this code specifically, so that generation of the vectors now uses code that has some noticeable relation to the compositional operations being performed. The next version was:

Parameters: int L, MDDL::mdd hatS, MDDL::mdd * T, MDDL::mdd * Tinv, int NQ, MDDL::mdd Bin, MDDL::OperatorCode cache

```
// calculate T2, T2inv, and NQ2, for lower levels
MDDL::mdd * T2 = T;
MDDL::mdd * T2inv = Tinv;
int NQ2 = NQ;
const int hatSL = hatS.level();
const int SXSL = hatSL+hatSL;

while( NQ2 && (T2[0].level()==SXSL)){ T2++; T2inv++; NQ2--; };

// variables for evaluating B\((U2/3 (T X hatS)\(_ X BoTinv)) U (U2/3 (hatS X Tinv)\(ToB X _)))
// == B\({(p q)| exists p': (p T p') /\ (q in hatS) /\ ~(exists q': p' B q' /\ q' Tinv q)} U {(p q)| exists q': (p in hatS) /\ (q' Tinv q) /\ ~(XXXexists p': p B p' /\ p' Tinv q
static Symbolic_Variable p("p",L);
static Symbolic_Variable pprime("pprime",L);
static Symbolic_Variable q("q",L);
static Symbolic_Variable qprime("qprime",L);

// B o Tinv // == {(p' q)| exists q': p' B q' /\ q' Tinv q}
static composition_spec OP_BoTinv(*this); // should be same as OP_ToB
OP_BoTinv.initialize2( (bool1&bool0), ( pprime % qprime % q ), USE()<< pprime << qprime, USE()<< qprime << q, PROJECT()<< qprime );

// T o B // == {(p q')| exists p': p T p' /\ p' B q'}
static composition_spec OP_ToB(*this); // should be same as OP_BoTinv
OP_ToB.initialize2( (bool1&bool0), ( p % pprime % qprime ), USE()<< p << pprime, USE()<< pprime << qprime, PROJECT()<< pprime );

// U2/3 (T X hatS)\(_ X BoTinv) // == {(p q)| exists p': (p T p') /\ (q in hS) /\ ~(p' BoTinv q)}
static composition_spec OP_UTXhSminusBoTinv(*this);
OP_UTXhSminusBoTinv.initialize3( (bool2&bool1&~bool0), ( p % pprime % q ), USE()<< p << pprime, USE()<< q, USE()<< pprime << q, PROJECT()<< pprime );

// U2/3 (hatS X Tinv)\(ToB X _) // == {(p q)| exists q': (p in hatS) /\ (q' Tinv q) /\ ~(p ToB q')}
static composition_spec OP_hSXTinvminusToB(*this);
OP_hSXTinvminusToB.initialize3( (bool2&bool1&~bool0), ( p % qprime % q ), USE()<< p, USE()<< qprime << q, USE()<< p << qprime, PROJECT()<< qprime );

//convenience:
int L2 = L+L;
int L3 = L+L+L;

MDDL::mdd B = Bin;
MDDL::mdd Bold = B;

// first, saturate lower levels:
B = BiSat1Saturationint( L, hatS, T2, T2inv, NQ2, B, cache );

// then do fixed point calculation:
do {
    Bold = B;
    // operate at this level:
    // apply each transition relation at this level (T is sorted by descending Top level):
    for ( int a=0; (a<NQ) && (T[a].level() == SXSL); ++a ) {

        MDDL::mdd Ta = T[a];
        MDDL::mdd Tinva = Tinv[a];

        // B o Tinv // == {(p' q)| exists q': p' B q' /\ q' Tinv q}
        MDDL::mdd BoTinv = OP_BoTinv.execute_on4( L3, L2, NULL, NULL, B, Tinva, MDD_GCOMPOSE43_QQ );

        // T o B // == {(p q')| exists p': p T p' /\ p' B q'}
        MDDL::mdd ToB = OP_ToB.execute_on4( L3, L2, NULL, NULL, Ta, B, MDD_GCOMPOSE44_QQ );

        // U2/3 (T X hatS)\(_ X BoTinv) // == {(p q)| exists p': (p T p') /\ (q in hS) /\ ~(p' BoTinv q)}
        MDDL::mdd UTXhSminusBoTinv = OP_UTXhSminusBoTinv.execute_on4( L3, L2, NULL, Ta, hatS, BoTinv, MDD_GCOMPOSE45_QQ );

        // U2/3 (hatS X Tinv)\(ToB X _) // == {(p q)| exists q': (p in hatS) /\ (q' Tinv q) /\ ~(p ToB q')}
        MDDL::mdd hSXTinvminusToB = OP_hSXTinvminusToB.execute_on4( L3, L2, NULL, hatS, Tinva, ToB, MDD_GCOMPOSE46_QQ );

        MDDL::mdd DeltaBbar = MDDL::g_mddf.or_qq(UTXhSminusBoTinv,hSXTinvminusToB);

        // put the results into B

        B = MDDL::g_mddf.minus_qq( B, DeltaBbar );

        // saturate those results:
        { if(DeltaBbar) if(B) B = BiSat1Saturationint( L, hatS, T2, T2inv, NQ2, B, cache ); };
    };
} while ( Bold != B );

MDDL::g_mddf.cache_add( cache, Bin, B );  // memoize

return B;
```

In the improved code, there is still a block of code near the beginning for initializing vectors, but here the code is less amorphous, and the vectors are hidden within objects of type composition_spec. Each composition_spec supplies all the information (except for parameters, sizes, and caching information) needed by a composition operation used later. This makes both the vector construction code and the composition operator code more readable. There is still considerable distance between the code and the algorithm pseudocode, as the set-theoretic notation of the pseudocode are not directly supported in C++. Additionally, cumbersome characteristics of the existing library interface are obvious, such as the use of function names (MDDL::g_mddf.minus_qq) that depend on which kind of MDDs are being used.

C++ does provide template meta-programming, which can be used to generate complex efficient code at compile time, while allowing slight coding syntax improvement. I propose to implement a library and interface that generates efficient code from input code closer to pseudocode algorithms for MDD operations. With the proposed improvements, I hope to instead write the following code:

```
typedef TeDDy::tupleset stateset;

typedef TeDDy::interleaved<stateset,stateset> intstaterelation;

Parameters: int L, stateset hatS, intstaterelation * T, intstaterelation * Tinv, int NQ, intstaterelation Bin, TeDDy::CacheGroup caches

// calculate T2, T2inv, and NQ2, for lower levels
stateset * T2 = T;
stateset * T2inv = Tinv;
int NQ2 = NQ;
const int hatSL = hatS.level();
const int TL1
const int SXSL = hatSL+hatSL;


while( NQ2 && (T2[0].level()==SXSL)){ T2++; T2inv++; NQ2--; };

// variables for evaluating B\((U2/3 (T X hatS)\(_ X BoTinv)) U (U2/3 (hatS X Tinv)\(ToB X _)))
// == B\({(p q)| exists p': (p T p') /\ (q in hatS) /\ ~(exists q': p' B q' /\ q' Tinv q)} U {(p q)| exists q': (p in hatS) /\ (q' Tinv q) /\ ~(XXXexists p': p B p' /\ p' Tinv q
#TeDDy_4_Symbolic_Tuple_Names( p, p_prime, q, q_prime )

//convenience:
int L2 = L+L;
int L3 = L+L+L;

TeDDy::mdd B = Bin;
TeDDy::mdd Bold = B;

// first, saturate lower levels:
B = BiSat1Saturationint( L, hatS, T2, T2inv, NQ2, B, caches );

// then do fixed point calculation:
do {
    Bold = B;
    // operate at this level:
    // apply each transition relation at this level (T is sorted by descending Top level):
    for ( int a=0; (a<NQ) && (T[a].level() == SXSL); ++a ) {

        TeDDy::mdd Ta = T[a];
        TeDDy::mdd Tinva = Tinv[a];

        // B o Tinv // == {(p' q)| exists q': p' B q' /\ q' Tinv q}
        TeDDy::mdd BoTinv = tuples2<intstaterelation>(p_prime,q)(exists(q_prime)( B(p_prime,q_prime) & Tinva(q_prime,q) ));
        BoTinv.withcache(caches).calculate();

        // T o B // == {(p q')| exists p': p T p' /\ p' B q'}
        TeDDy::mdd ToB = tuples2<intstaterelation>(p,q_prime)(exists(p_prime)(Ta(p,p_prime) & B(p_prime,q_prime)));
        ToB.withcache(caches).calculate();

        // U2/3 (T X hatS)\(_ X BoTinv) // == {(p q)| exists p': (p T p') /\ (q in hS) /\ ~(p' BoTinv q)}
        TeDDy::mdd UTXhSminusBoTinv = tuples2<intstaterelation>(p,q)(exists(p_prime)(Ta(p,p_prime) & hatS(q) & ~BoTinv(p_prime,q)));
        UTXhSminusBoTinv.withcache(caches).calculate();

        // U2/3 (hatS X Tinv)\(ToB X _) // == {(p q)| exists q': (p in hatS) /\ (q' Tinv q) /\ ~(p ToB q')}
        TeDDy::mdd hSXTinvminusToB = tuples2<intstaterelation>(p,q)(exists(q_prime)(hatS(p) & Tinva(q_prime,q) & ~ToB(p,q_prime)));
        hSXTinvminusToB.withcache(caches).calculate();

        TeDDy::mdd DeltaBbar = UTXhSminusBoTinv .U hSXTinvminusToB;

        // put the results into B

        B = ( B - DeltaBbar ).withcache(caches).calculate();

        // saturate those results:
        { if(DeltaBbar) if(B) B = BiSat1Saturationint( L, hatS, T2, T2inv, NQ2, B, caches ); };
    };
} while ( Bold != B );

caches.add( &BiSatSaturationint, Bin, B );  // memoize

return B;
```

Note first that this code should not be taken as a specific definitive example, and that some library syntax and design issues remain open for resolution in the research phase of this project. In this code, there is no longer a separate block of code for initializing the control vectors for another block of code. Instead, the main block of code invokes a customized set of recursive functions which implicitly hold the control knowledge previously stored in the control vectors. Additional advantages of this improved coding style are obvious, including the following:

1. The use of types to indicate various forms of encoding, such as interleaved pairs vs. concatenated pairs.
2. The use of a single name to denote a given operator, independent of which type of tuple sets and reductions are used in the encoding.
3. The simple combination of multiple operators into efficient aggregate operations.
4. The use of comprehension-like structures, having local variable names, to define relations.

3.3 Locality enhancement for LTS for weak bisimulation

My saturation-based bisimulation algorithm in Section 2.4.2 is weak in the cases where there are transition relations having large support. I expect that in many of these cases, such a transition relation may have useful projections onto *uncertain transition relation*s with relatively small support. An uncertain transition relation is a transition relation where some pairs in the relation have an imprecise specification for either (or both) the domain element or (or and) the range element. Such a projection is not necessarily lossless. I hope to automatically decompose transition relations with large support into multiple uncertain transition relations having small support. For the purpose of calculating $\sim$, both the original transition relations and the decomposed uncertain transition relations would be used. In the process of calculating $\sim$, the saturation based algorithm would give priority to use of the decomposed uncertain transition relations (as they have small support, hence expected lower usage cost) over the use of the original transition relations having large support. The hope is that, as with many other saturation-based algorithms, most of the useful work would be performed by usage of the transitions relations having small support, leaving less need for using transitions relations having large support, so that the overall run-time would decrease, compared with the previous algorithm.

3.4 Fully-Symbolic lumping algorithm

I expect to be able to manipulate the definition of lumping, given in Section 2.4.3 into a form directly applicable to a saturation-based implementation, analogously to how the definition of bisimulation was manipulated in Section 2.4.2. If I succeed in this endeavor, the resulting algorithm would be (nearly) the first fully symbolic lumping algorithm. The algorithm in [16](§6.49) is fully symbolic, but in a way that is somewhat arbitrary and unnatural. It is likely that such an algorithm would be the only feasible way to perform lumping on systems where there are very many equivalence classes.

3.5 Novel speculation heuristics for parallel GDD library

This research presents another opportunity to attempt to obtain the parallel scalability gains long hoped for by model checking researchers. The following techniques have not yet been explored, yet appear to present obvious opportunities for performance enhancement through parallelism. Note that these descriptions below are merely summaries of the most important parts, made necessarily brief due to the impending

deadlines, and in no way illustrate the full extent of my thinking on this subject. The first three speculative techniques relate to library-level parallelism and could be implemented with the TeDDy library, while the final technique relates more specifically to saturation-based algorithms, especially for the purpose of state-space exploration. Fundamental library operations on DDs involve traversals of homologous parts of each DD input involved in the operation. Taking the union, for example, of A and B involves coordinated traversal of homologous parts of A and B, invoking the union operation on homologous children (and further descendants) of A and B. A primary problem with the 'distribution by level' parallelism scheme described in Section 2.5.1 is that traversal requests sent from the processor holding a parent node to the processor holding the child nodes of that parent all arrive together, presenting plenty of work to the processor holding the child nodes, but not to any other processors, limiting the spread of parallel activity. These first three techniques speculatively initiate parallel activity at lower levels.

3.5.1 Forward speculation

As can be understood from the *union*() algorithm in Section 2.1.2, the choice of which descendants of A and B are to be combined in a nested *union*() operations cannot be reliably determined a priori. *Forward speculation* chooses some remote descendants of A and B based on information cached with A and B, possibly including a list of 'preferred' descendants, and always at a specific level. Thus, the *union*() function applied to A and B will speculatively invoke *union*() function applied to some pairs of various remote descendants of A with various remote descendants of B, potentially producing results which will be used in the construction of $union(A, B)$. In the case where the level of chosen descendants actually has very few descendants of A and/or B, this method is more likely to produce useful speedup.

3.5.2 Reverse speculation

Without speculation, (at lower levels of DD) $union(A, B)$ is calculated only if it is a necessary part of a higher-level *union*() operation where A and B are homologous components of operands in the higher-level *union*() operation. *Reverse speculation* speculatively calculates $union(A, B)$ based on information cached with A and B, possibly including information about the sequence of labels on paths which lead to A and to B. Such information might be stored in compressed form as a signature. Thus, when a higher-level *union*() is requested, a speculative calculation of $union(A, B)$ may be launched if a signature of a path leading to A matches a signature of a path leading to B. This technique seems more likely to produce a useful speedup in cases where there are few paths leading to A and/or to B, allowing for efficient search for matching path signatures.

3.5.3 Count-based speculation

Count-based speculation considers all levels of the operands of a highest-level DD operation, to find levels where the upper bound on the number of output nodes is lowest. If the bound on the number of output nodes at a given level is below some carefully tuned threshold, speculative calculation of possible output nodes at that level is initiated. Aside from the cost of extra bookkeeping, this technique costs very little when the threshold is very low, although it may also be less likely to produce useful speedup.

3.6 Re-organizing parallel saturation

The efficiency of saturation-based state-space exploration, and many other DD algorithms, is strongly influenced by the necessary ordering of variables, and by the support of related transition relations, as abbreviated in the form of event spans. The *event span* of a transition relation (given a specific variable

ordering) is the smallest set of contiguous levels containing the support variables of the relation. For purposes of saturation-based state-space exploration, all other things being equal, a variable ordering which causes transition relations to have small event spans is preferred over a variable ordering which causes transition relations to have large event spans. Typically, before state-space exploration begins, the variables have been pre-ordered in such a way as to bring about reasonably small event spans when possible. This heuristic proposes to take further advantage of this ordering to identify opportunities for parallel firing of events with non-overlapping spans, during state-space exploration. Due to time limitations, I will simplify this description to the case where only two processors are available. Sequential saturation-based state-space exploration consists of: initializing a DD S with an encoding of the set of initial system states, followed by augmentation of S through firing individual events (transition relations), in the order indicated by the saturation heuristic, after which S contains an encoding of the entire reachable state-space. The saturation heuristic gives priority to events with a lower *top*, where the top of an event is the highest level of its event span. This heuristic divides the levels into two groups (high and low) and events into three groups (high, low, and hybrid). Each group (high and low) of levels is contiguous and comprises about half of the groups, where the exact point of division between them may be subject to tuning. The high group of events comprises events where the span is entirely within the high levels. Analogously, the low group of events comprises events where the span is entirely within the low levels. The remaining events comprise the hybrid group, which is likely to be small given a good variable ordering. However, any system with no hybrid events could simply be factored into multiple independent systems, hence realistic systems will always have at least one hybrid event. This technique alters the saturation order by making the high and low group of events independent of each other, while the high and low groups of events both have priority over the hybrid events. Within the DD encoding S, the nodes corresponding to high levels are separated from the nodes corresponding to the low levels by an extra invisible level of symbolic encoding, which allows concurrent firing of high events and low events, each operation only on the corresponding levels of S. When high and low events are no longer able to fire due to (possibly temporary) convergence of S, The extra level of symbolic encoding is processed to unify the higher and lower levels of S into a single DD, after which hybrid events may fire. When it is time to attempt firings of high and/or low events, the invisible level of symbolic encoding is re-imposed prior to such firings. The advantage of this scheme is that high events may be fired in parallel with low events, and there should be very few hybrid events. The disadvantages are that the invisible level of symbolic encoding must be processed, possibly many times, before proceeding, and that hybrid events must proceed sequentially. These disadvantages have not been quantified, yet their extent may (or may not) dominate the benefits of this technique.

4 Options and future work

4.1 Additional encoding to promote sharing

As footnoted after rule 6'b in Section 3.1.1, rule 6 may not be the optimal rule to chose for purposes of promoting maximal sharing among sub-trees in a GDD. It is possible, however, to potentially increase sharing by using an additional layer of encoding for variable names (references to members of the tuple argument of an encoded function). This additional layer of encoding may be thought of as renamings of variables occurring in a sub-tree, applied whenever the subtree is accessed by a specific edge, which will have an annotation giving the renaming. It seams that relatively few situations occur where sharing will be improved by such an additional layer of encoding, however, this has not been quantified, and so could be a subject of future research.

4.2 Overhead improvement for symbolic methods

As noted in Section 2.7, there is overhead proportional to tuple size, when using unreduced decision diagrams to encode tuple-sets. It is desirable to find some way to decrease this overhead so that it is never onerous. A careful examination of the example in Section 2.7 shows that a constant-reduced encoding (where the constant is 1) reduces the BDD encoding of $\{\langle 1,0,1,1,0,1\rangle\}$ to 2 non-leaf nodes, when null-pointer elimination is also used. Although such a reduction will typically produce a factor of 2 in space usage, it is desirable to encode sets using a number of nodes linear in the size of the set, so that DD encodings could, at least theoretically, compete with element list encodings.

The condition that makes a DD encoding (sometimes) occupy more nodes than the corresponding list encoding, is that some DD nodes have only a single (non-leaf) child. If every DD node of a K-level DD (encoding a set of K-tuples) had 2 children, the DD would have $2^K - 1$ nodes encoding a set of 2^K K-tuples, so that the overhead would be entirely reasonable. What is needed then, is a special encoding for a sequence of nodes having only a single child. Understanding my proposed solution to this problem requires noticing that, the explicit encoding is more compact only when many parts (or all) of the tuple may be encoded compactly as a string of a few bits (up to about the size of a pointer).

My proposed solution (for MDDs) is to condense a sequence of nodes having only a single child into an additional annotation on an edge. Such an annotation contains a compact representation of the sequence of tuple element values associated with that sequence of nodes. Thus, every node that has only one child is condensed to part of an annotation on an edge, so that all remaining nodes have at least 2 children each. This adjustment results in MDDs that encode sets with at least as many members as there are nodes in the encoding.

4.3 Automatic reduction selection per edge

One disadvantage with the use of various reduction schemes is that, in current DD libraries, the reduction scheme for each DD must be chosen by the programmer using knowledge of the expected structure of the encoded set. Typically, an entire DD must be encoded using a single reduction scheme, so that different branches of a DD must be encoded using the same reduction scheme. However, the necessity of manual selection of reductions may be removed, and the flexibility of branch-specific reductions may be added, through an adjustment of the scheme described in Section 4.2.

In this scheme, within a GDD, any node with only 1 non-leaf child, and at most 1 other (leaf) child would be condensed to part of an edge annotation, and such annotations would be additionally condensed in cases where a simple pattern repeats many times, as would be reduced in fully or fully-identity reduced GDDs. Such an encoding scheme has the benefit of automatic condensation of GDD nodes when any of the previous reductions would apply, without the necessity of manual selection. Thus, this potential feature would provide the benefits of all extant reduction techniques, without additional effort on the part of the library user.

In more distant future work, an additional step could be taken, allowing the use of some more general data compression method to be applied to sequences of simple nodes. However, schedule limitations may place such work beyond the scope of the proposed research.

5 Schedule

Here I first enumerate the tasks involved in this research, and then propose a specific schedule for their execution.

5.1 Tasks

I propose the following plans for technical tasks be performed prior to dissertation, in addition to solving any further research problems that arise within these tasks. These plans include two options, Plan C, and Plan D which is now preferred due to the 10-year time limit imposed on my scholastic duration by the Graduate Division[2]. Additionally, I prefer to defer any writing tasks (such as papers and other correspondence) until after the technical tasks are finished and the dissertation is started, so that enough research may be accomplished to support the dissertation.

5.1.1 Plan C

Plan C is technologically more interesting, however it is no longer preferred, due to schedule constraints. It is based on the expectation of starting the technical tasks in January 2014. Plan C comprises the following tasks, and the schedule for Plan C is shown in Figure 7.

1. Implement GDDs in TeDDy library.

   The functionality of GDD operations, with possible additional improvements, will be implemented as a library, with a well defined (but possibly overly verbose) API. This must also be done keeping in mind the later parallel implementation, as well as desired properties of the user library interface.

   Basic functionality improvements shall be:

   (a) GDD set operations with 0-6 GDD parameters with single quantification and result caching.

   (b) Checking of match between interleaved/non-interleaved parameter types and corresponding number of levels.

   Additional or delayed improvements might be as follows:

   (a) Edge-Valued diagrams with sum quantification for EV+GDDs and sum and product quantification for EV*GDDs

   (b) Compression of node contents for common patterns and/or small indices

   (c) Additional layer of encoding to improve sharing (Section 4.1)

   (d) Fragmentation-proof memory allocation and/or variation of space reclamation methods

   (e) Automatic reduction choice (Section 4.3)

   (f) Per-operation memo-cache designation and/or cache groups

   (g) Access by level

   (h) Access by variable

   (i) Per-DD pseudo-forrest designation

2. Construction of novel library interface.

   One or more layers of API will be added to the GDD library, to allow user coding of relatively elegant yet efficient model checking code. Assuming the language for the project is C++, this task will primarily involve template metaprogramming as described in Section 3.2.

[2] Plans A and B [36](§6.48) have been abandoned, as being too aggressive for the time remaining after the upcoming proposal defense.

3. Demonstration of improved library.

   I will re-implement and analyze the bisimulation algorithms from our paper [38](§6.3), including my weak bisimulation algorithm (Section 2.4.2) and Wimmer's bisimulation algorithm [54](§6.54) using the new GDD library.

4. Locality enhancement for LTS for weak bisimulation.

   I will attempt to implement this improvement to fully symbolic bisimulation, as described in Section 3.3, using the new GDD library.

5. Fully Symbolic Lumping.

   I will attempt to implement a novel fully symbolic lumping algorithm, following the plan described in Section 3.4, using the new GDD library.

6. Parallel implementation of TeDDy, including GDDs and improved interface.

   I will choose a suitable platform, which must be a platform that offers scaling to at least 1K 'large' processors (CISC with >1G RAM each), or 1M 'tiny' processors, in a system available to me. I will then port our new GDD library to the chosen platform, probably using one or more of the speculation techniques described in Section 3.5. This may require an initial careful study of the communication infrastructure of the target platform.

7. Parallel Saturation Algorithm based on model locality.

   Based on the above parallel library implementation, I will attempt to implement parallel saturation-based state-space exploration using the re-organized saturation scheme described in Section 3.6.

8. Practical Application.

   A relevant model-checking application will be chosen and implemented using the new parallel GDD library and parallel saturation-based state-space exploration.

As opportunities arise, it will be advantageous to also perform some of the following publishing-related activities:

PA: Paper on the properties of GDDs.

PB: Report on the TeDDy with GDDs and the novel interface.

PC: Paper on Artificial Locality Enhancement, or Using Uncertainty to Improve Locality.

PD: Paper on Saturation-based Fully Symbolic Lumping.

PE: Papers on Parallel TeDDy implementation, and related parallel library algorithms.

PF: Papers on Model-Locality based parallel Saturation implementation.

PG: Papers on any additional research problems solved.

Some of the following additional activities will be necessary:

H: Writing and defending this proposal.

J: Apply for funding from NSF, or other sources.

K: Write Dissertation.

L: Defend Dissertation.

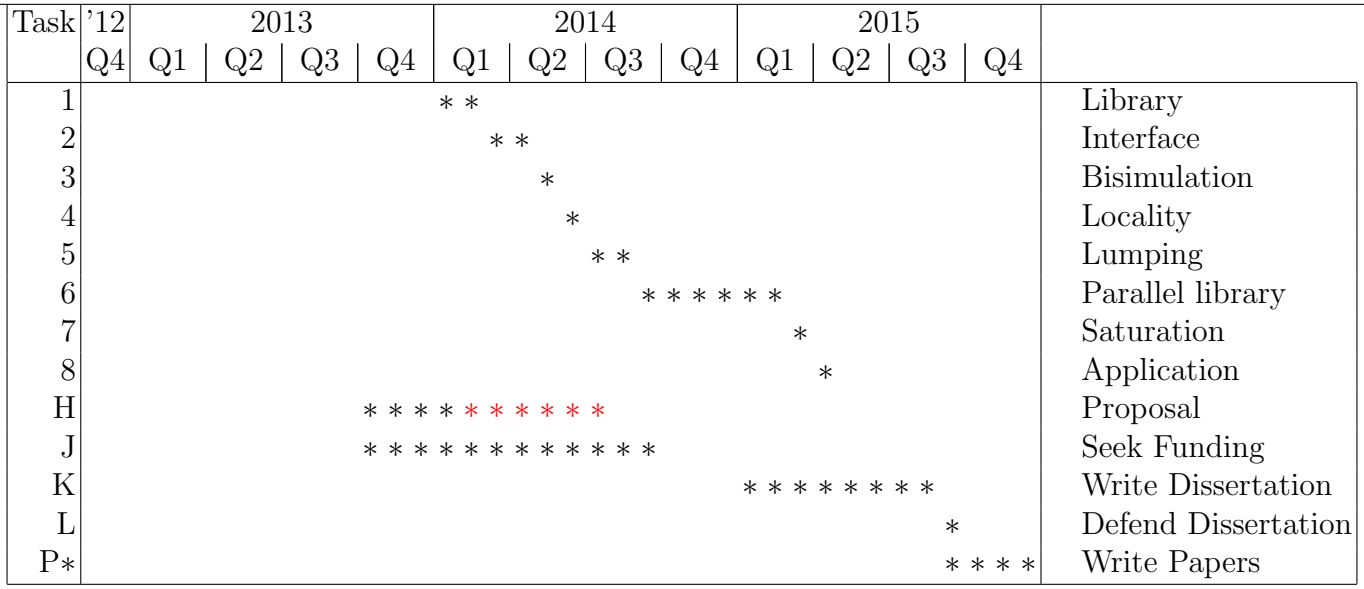


Fig. 7: Schedule (Plan C), (not preferred, due to amount of time used up on writing proposals)

5.1.2 Plan D

Due to scheduling constraints, I currently prefer this plan. Its description is identical to Plan C except for item 6, and the details of the schedule. Plan D comprises the following tasks, and the schedule for Plan D is shown in Figure 8.

1. Implement GDDs in TeDDy library.

   This task description is the same as in Plan C.

2. Construction of novel library interface.

   This task description is the same as in Plan C.

3. Demonstration of improved library.

   This task description is the same as in Plan C.

4. Locality enhancement for LTS for weak bisimulation.

   This task description is the same as in Plan C.

5. Fully Symbolic Lumping.

   This task description is the same as in Plan C.

6. Parallel implementation of TeDDy, including GDDs and improved interface.

   I will choose a suitable platform, which must be a roughly-symmetric parallel processor that offers scaling to at least 12 cores, with a current implementation of at least 4 cores, in a system available to me. I will then port our new GDD library to the chosen platform, probably using one or more of the speculation techniques described in Section 3.5. This may require an initial careful study of the thread library of the target platform.

7. Parallel Saturation Algorithm based on model locality.

   This task description is the same as in Plan C.

8. Practical Application.

   This task description is the same as in Plan C.

As opportunities arise, it will be advantageous to also perform some of the following publishing-related activities:

PA: Paper on the properties of GDDs.

PB: Report on the TeDDy with GDDs and the novel interface.

PC: Paper on Artificial Locality Enhancement, or Using Uncertainty to Improve Locality.

PD: Paper on Saturation-based Fully Symbolic Lumping.

PE: Papers on Parallel TeDDy implementation, and related parallel library algorithms.

PF: Papers on Model-Locality based parallel Saturation implementation.

PG: Papers on any additional research problems solved.

Some of the following additional activities will be necessary:

H: Writing and defending this proposal.

J: Apply for funding from NSF, or other sources.

K: Write Dissertation.

L: Defend Dissertation.

Task	'12	2013				2014				2015				
	Q4	Q1	Q2	Q3	Q4	Q1	Q2	Q3	Q4	Q1	Q2	Q3	Q4	
1								* * * *						Library
2								* * * *						Interface
3									*					Bisimulation
4									*					Locality
5									* *					Lumping
6								* * * *						Parallel library
7										*				Saturation
8										*				Application
H					* * * * * * * * * *									Proposal
J								* * *						Seek Funding
K										* * * * * * * *				Write Dissertation
L												*		Defend Dissertation
P*												* * * *		Write Papers

Fig. 8: Schedule (Plan D)

6 Annotated Bibliography

I considered many papers from many conference proceedings, and many journals. After reading the abstracts (and frequently much more) of the many chosen papers, the approximately 55 papers and other publications discussed in this section were selected as being possibly relevant to the current proposal.

The publications fall into several categories:

1. Publications that relate to the theory of decision diagrams, model checking and related algorithms.
2. Publications that illustrate methods that provide scalable parallelism for some applications, that could hint at techniques that could be useful for parallel implementation of TeDDy.
3. Finally, publications that must be included in such proposals, although practically irrelevant.

I have attempted to list the publications in order corresponding to the above list of categories, although some publications fall into more than one of these categories. Each entry references the bibliography at the end of this document for full bibliographic information. Each entry also includes a link, usually to the relevant DOI page, so the reader may easily access the original works. In some cases there is also a link to an on-line copy of the publication itself. In one notable case [49](§6.32), there is a link to the on-line video of the presentation.

6.1 A fine-grained fullness-guided chaining heuristic for symbolic reachability analysis [10]

Due to time constraints, I merely copy the abstract here:

> Chaining can reduce the number of iterations required for symbolic state-space generation and model-checking, especially in Petri nets and similar asynchronous systems, but requires considerable insight and is limited to a static ordering of the events in the high-level model.

We introduce a two-step approach that is instead fine-grained and dynamically applied to the decision diagrams nodes. The first step, based on a precedence relation, is guaranteed to improve convergence, while the second one, based on a notion of node fullness, is heuristic. We apply our approach to traditional breadth-first and saturation state-space generation, and show that it is effective in both cases.

DOI: http://dx.doi.org/10.1007/11901914_7

6.2 A Fully Symbolic Bisimulation Algorithm [37]

Due to time constraints, I merely copy the abstract here:

We apply the saturation heuristic to the bisimulation problem for deterministic discrete-event models, obtaining the fastest to date symbolic bisimulation algorithm, able to deal with large quotient spaces. We compare performance of our algorithm with that of Wimmer et al., on a collection of models. As the number of equivalence classes increases, our algorithm tends to have improved time and space consumption compared with the algorithm of Wimmer et al., while, for some models with fixed numbers of state variables, our algorithm merely produced a moderate extension of the number of classes that could be processed before succumbing to state-space explosion. We conclude that it may be possible to solve the bisimulation problem for systems having only visible deterministic transitions (e.g., Petri nets where each transition has a distinct label) even if the quotient space is large (e.g., 10^9 classes), as long as there is strong event locality.

DOI: http://dx.doi.org/10.1007/978-3-642-24288-5_19

6.3 AN EFFICIENT FULLY SYMBOLIC BISIMULATION ALGORITHM FOR NON-DETERMINISTIC SYSTEMS [38]

Due to time constraints, I merely copy the abstract here:

The definition of bisimulation suggests a partition-refinement step, which we show to be suitable for a saturation-based implementation. We compare our fully symbolic saturation-based implementation with the fastest extant bisimulation algorithms over a set of benchmarks, and conclude that it appears to be the fastest algorithm capable of computing the largest bisimulation over very large quotient spaces.

DOI: http://dx.doi.org/10.1142/S012905411340011X

6.4 Achieving Scalability in Parallel Reachability Analysis of Very Large Circuits [28]

Due to time constraints, I merely copy the abstract here:

This paper presents a scalable method for parallel symbolic reachability analysis on a distributed-memory environment of workstations. our method makes use of an adaptive partitioning algorithm which achieves high reduction of space requirements. The memory balance is maintained by dynamically repartitioning the state space throughout the computation. A

compact BDD representation allows coordination by shipping BDDs from one machine to another, where different variable orders are allowed. The algorithm uses a distributed termination protocol which none of the memory modules preserving a complete image of the set of reachable states. No external storage is used not the disk; rather, we make use of the network which is much faster. We implemented our method on a standard, loosely-connected environment of workstations, using a high-performance model checker. Our initial performance evaluation using several large circuits shows that our method can handle models that are too large to fit in the memory of a single node. The efficiency of the partitioning algorithm is linear in the number of workstations employed, with a 40-60% efficiency. A corresponding decrease of space requirements is measured throughout the reachability analysis. Our results show that the relatively-slow network does not become a bottleneck, and that computation time is kept reasonably small.

link: http://www.cs.technion.ac.il/users/orna/CAV00-scalable.ps

6.5 A Work-Efficient Distributed Algorithm for Reachability Analysis [26]

Due to time constraints, I merely copy the abstract here:

This work presents a novel distributed, symbolic algorithm for reachability analysis that can effectively exploit, as needed, a large number of machines working in parallel. The novelty of the algorithm is in its dynamic allocation and reallocation of processes to tasks and in its mechanism for recovery, from local state explosion. As a result, the algorithm is work-efficient: it utilizes only those resources that are actually needed. In addition, its high adaptability makes it suitable for exploiting the resources of very large and heterogeneous distributed, non-dedicated environments. Thus, it has the potential of verifying very large systems. We implemented our algorithm in a tool called Division. Our preliminary experimental results show that the algorithm is indeed work-efficient. Although that the goal of this research is to check larger models, the results also indicate the potential to obtain high speedups, because communication overhead is very small.

DOI: http://dx.doi.org/10.1007/978-3-540-45069-6_5

6.6 Scalable Distributed On-The-Fly Symbolic Model Checking [3]

Due to time constraints, I merely copy the abstract here:

This paper presents a scalable method for parallel symbolic on-the- fly model checking on a distributed-memory environment of workstations. Our method combines a parallel version of an on-the-fly model checker for safety properties with a scalable scheme for reachability analysis. The extra load of storage required for counter example generation is evenly distributed among the processes by our memory balancing. For the sake of scalability, at no point during computation the memory of a single process contains all the data from any of the cycles. The counter example generation is thus performed through collaboration of the parallel processes. We develop a method for the counter example generation keeping a low peak memory requirement during the backward step and the computation of the inverse transition relation. We implemented our method on a standard, loosely-connected environment of workstations, using a high-performance SMV-based model checker. Our initial performance evaluation using several large circuits shows that our method can check models that are too large to fit in the

memory of a single node. Our on-the-fly approach may find counter examples even when the model is too large to fit in the memory of the parallel system.

DOI: http://dx.doi.org/10.1007/3-540-40922-X_24

6.7 Achieving Speedups in Distributed Symbolic Reachability Analysis through Asynchronous Computation [25]

Due to time constraints, I merely copy the abstract here:

This paper presents a novel BDD-based distributed algorithm for reachability analysis which is completely asynchronous. Previous BDD-based distributed schemes are synchronous: they consist of interleaved rounds of computation and communication, in which the fastest machine (or one which is lightly loaded) must wait for the slowest one at the end of each round. We make two major contributions. First, the algorithm performs image computation and message transfer concurrently, employing non-blocking protocols in several layers of the communication and the computation infrastructures. As a result, regardless of the scale and type of the underlying platform, the maximal amount of resources can be utilized efficiently. Second, the algorithm incorporates an adaptive mechanism which splits the workload, taking into account the availability of free computational power. In this way, the computation can progress more quickly because, when more CPUs are available to join the computation, less work is assigned to each of them. Less load implies additional important benefits, such as better locality of reference, less overhead in compaction activities (such as reorder), and faster and better workload splitting. We implemented the new approach by extending a symbolic model checker from Intel. The effectiveness of the resulting scheme is demonstrated on a number of large industrial designs as well as public benchmark circuits, all known to be hard for reachability analysis. Our results show that the asynchronous algorithm enables efficient utilization of higher levels of parallelism. High speedups are reported, up to an order of magnitude, for computing reachability for models with higher memory requirements than was previously possible.

DOI: http://dx.doi.org/10.1007/11560548_12

6.8 Verifying Very Large Industrial Circuits Using 100 Processes and Beyond [22]

Due to time constraints, I merely copy the abstract here:

Recent advances in scheduling and networking have cleared the way for efficient exploitation of large-scale distributed computing platforms, such as computational grids and huge clusters. Such infrastructures hold great promise for the highly resource-demanding task of verifying and checking large models, given that model checkers would be designed with a high degree of scalability and flexibility in mind. In this paper we focus on the mechanisms required to execute a high-performance, distributed, symbolic model checker on top of a large-scale distributed environment. We develop a hybrid algorithm for slicing the state space and dynamically distribute the work among the worker processes. We show that the new approach is faster, more effective, and thus much more scalable than previous slicing algorithms. We then present a checkpoint-restart module that has very low overhead. This module can be used to combat failures which become probable with the size of the computing platform. However, checkpoint-restart is even

more handy for the scheduling system: it can be used to avoid reserving large numbers of workers, thus making the distributed computation work-efficient. Finally, we discuss for the first time the effect of reorder on the distributed model checker and show how the distributed system performs more efficient reordering than the sequential one. We implemented our contributions on a network of 200 processors, using a distributed scalable scheme that employs a high-performance industrial model checker from Intel. Our results show that the system was able to verify real-life models much larger than was previously possible.

DOI: http://dx.doi.org/10.1007/11562948_4

6.9 Roomy: A System for Space Limited Computations [31]

Due to time constraints, I merely copy the abstract here:

There are numerous examples of problems in symbolic algebra in which the required storage grows far beyond the limitations even of the distributed RAM of a cluster. Often this limitation determines how large a problem one can solve in practice. Roomy provides a minimally invasive system to modify the code for such a computation, in order to use the local disks of a cluster or a SAN as a transparent extension of RAM.

Roomy is implemented as a C/C++ library. It provides some simple data structures (arrays, unordered lists, and hash tables). Some typical programming constructs that one might employ in Roomy are: map, reduce, duplicate elimination, chain reduction, pair reduction, and breadth-first search. All aspects of parallelism and remote I/O are hidden within the Roomy library.

DOI: http://dx.doi.org/10.1145/1837210.1837216

6.10 Parallel Disk-Based Computation for Large, Monolithic Binary Decision Diagrams [32]

Due to time constraints, I merely copy the abstract here:

Binary Decision Diagrams (BDDs) are widely used in formal verification. They are also widely known for consuming large amounts of memory. For larger problems, a BDD computation will often start thrashing due to lack of memory within minutes. This work uses the parallel disks of a cluster or a SAN (storage area network) as an extension of RAM, in order to efficiently compute with BDDs that are orders of magnitude larger than what is available on a typical computer. The use of parallel disks overcomes the bandwidth problem of single disk methods, since the bandwidth of 50 disks is similar to the bandwidth of a single RAM sub-system. In order to overcome the latency issues of disk, the Roomy library is used for the sake of its latency-tolerant data structures. A breadth-first algorithm is implemented. A further advantage of the algorithm is that RAM usage can be very modest, since its largest use is as buffers for open files. The success of the method is demonstrated by solving the 16-queens problem, and by solving a more unusual problem — counting the number of tie games in a three-dimensional 4x4x4 tic-tac-toe board.

DOI: http://dx.doi.org/10.1145/1837210.1837222

6.11 Distributed Saturation [8]

Due to time constraints, I merely copy the abstract here:

> The Saturation algorithm for symbolic state-space generation, has been a recent breakthrough in the exhaustive verification of complex systems, in particular globally-asynchronous/locally-synchronous systems. The algorithm uses a very compact Multiway Decision Diagram (MDD) encoding for states and the fastest symbolic exploration algorithm to date. The distributed version of Saturation uses the overall memory available on a network of workstations (NOW) to efficiently spread the memory load during the highly irregular exploration. A crucial factor in limiting the memory consumption during the symbolic state-space generation is the ability to perform *garbage collection* to free up the memory occupied by dead nodes. However, garbage collection over a NOW requires a nontrivial communication overhead. In addition, operation cache policies become critical while analyzing large-scale systems using the symbolic approach. In this technical report, we develop a garbage collection scheme and several operation cache policies to help on solving extremely complex systems. Experiments show that our schemes improve the performance of the original distributed implementation, SmArTNow, in terms of time and memory efficiency.

URL: `http://ntrs.nasa.gov/search.jsp?R=20070017995`

6.12 Caching, Hashing, and Garbage Collection for Distributed State Space Construction [9]

Due to time constraints, I merely copy the abstract here:

> The Saturation algorithm for symbolic state-space generation is a recent advance in exhaustive verification of complex systems, in particular globally-asynchronous/ locally-synchronous systems. The distributed version of Saturation uses the overall memory available on a network of workstations (NOW) to efficiently spread the memory load during its highly irregular exploration. A crucial factor in limiting the memory consumption in symbolic state-space generation is the ability to perform garbage collection to free up the memory occupied by dead nodes. However, garbage collection over a NOW requires a nontrivial communication overhead. In addition, operation cache policies become critical while analyzing large-scale systems using a symbolic approach. In this paper, we develop a garbage collection scheme and several operation cache policies to help the analysis of complex systems. Experiments show that our schemes improve the performance of the original distributed implementation, SmartNOW, in terms of both time and memory efficiency.

URL: `http://www.cs.ucr.edu/~ciardo/pubs/2007PDMC-DistSMART.pdf`

6.13 Parallelising symbolic state-space generators [17]

Due to time constraints, I merely copy the abstract here:

> Symbolic state-space generators are notoriously hard to parallelise, largely due to the irregular nature of the task. Parallel languages such as Cilk, tailored to irregular problems, have been shown to offer efficient scheduling and load balancing. This paper explores whether Cilk

can be used to efficiently parallelise a symbolic state-space generator on a shared-memory architecture. We parallelise the Saturation algorithm implemented in the SMART verification tool using Cilk, and compare it to a parallel implementation of the algorithm using a thread pool. Our experimental studies on a dual-processor, dual-core PC show that Cilk can improve the run-time efficiency of our parallel algorithm due to its load balancing and scheduling efficiency. We also demonstrate that this incurs a significant memory overhead due to Cilk's inability to support pipelining, and conclude by pointing to a possible future direction for parallel irregular languages to include pipelining.

DOI: http://dx.doi.org/10.1007/978-3-540-73368-3_31

6.14 Parallel symbolic state-space exploration is difficult, but what is the alternative? [13]

Due to time constraints, I merely copy the abstract here:

State-space exploration is an essential step in many modeling and analysis problems. Its goal is to find the states reachable from the initial state of a discrete-state model described. The state space can used to answer important questions, e.g., "Is there a dead state?" and "Can N become negative?", or as a starting point for sophisticated investigations expressed in temporal logic. Unfortunately, the state space is often so large that ordinary explicit data structures and sequential algorithms cannot cope, prompting the exploration of (1) parallel approaches using multiple processors, from simple workstation networks to shared-memory supercomputers, to satisfy large memory and runtime requirements and (2) symbolic approaches using decision diagrams to encode the large structured sets and relations manipulated during state-space generation. Both approaches have merits and limitations. Parallel explicit state-space generation is challenging, but almost linear speedup can be achieved; however, the analysis is ultimately limited by the memory and processors available. Symbolic methods are a heuristic that can efficiently encode many, but not all, functions over a structured and exponentially large domain; here the pitfalls are subtler: their performance varies widely depending on the class of decision diagram chosen, the state variable order, and obscure algorithmic parameters. As symbolic approaches are often much more efficient than explicit ones for many practical models, we argue for the need to parallelize symbolic state-space generation algorithms, so that we can realize the advantage of both approaches. This is a challenging endeavor, as the most efficient symbolic algorithm, Saturation, is inherently sequential. We conclude by discussing challenges, efforts, and promising directions toward this goal.

DOI: http://dx.doi.org/10.4204/EPTCS.14.1

6.15 Implementation of an Efficient Parallel BDD Package [46]

Due to time constraints, I merely copy the abstract here:

DOI: http://dx.doi.org/10.1145/240518.240639

6.16 AN ANTICIPATED FIRING SATURATION ALGORITHM FOR SHARED-MEMORY ARCHITECTURES [20]

Due to time constraints, I merely copy the abstract here:

Parallelising symbolic state-space generation algorithms, such as Saturation, is known to be difficult as it often incurs high parallel overheads. To improve efficiency, related work on a distributed-memory implementation of Saturation proposed using idle processors for speculatively firing events and caching the obtained results, in the hope that these results will be needed lateron. This paper investigates a variant of this anticipated firing approach for shared-memory architectures, such as multi-core PCs. Rather than parallelising Saturation, the idea is to run the sequential Saturation algorithm on one core, while the others are given speculative work. Since computing the optimal strategy for selecting useful work is likely to be an NP-complete problem, the paper devises and implements various heuristics. The obtained experimental results show that moderate speed-ups can be achieved as a result of using anticipated firing. However, the proposed heuristics require further work in order to be truly useful in practice.

link: http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.96.7255

6.17 To Parallelize or to Optimize? [19]

Due to time constraints, I merely copy the abstract here:

Model checking is a popular and successful technique for verifying complex digital systems. Carrying this techniqueand its underlying state-space generation algorithmsbeyond its current limitations presents itself with a number of alternatives. Our focus is on parallelization which is made attractive by the current trend in hardware architectures towards multi-core, multiprocessor systems. The main obstacle in this endeavour is that, in particular, symbolic state-space generation algorithms are notoriously hard to parallelize. In this article, we describe the process of taking a sequential symbolic state-space generation algorithm, namely a generic, symbolic BFS algorithm, through a sequence of optimizations that leads up to the Saturation algorithm and follow the impact these sequential algorithms have on their parallel counterparts. In particular, we develop a parallel version of Saturation, discuss the challenges faced in its design and conduct extensive experimental studies of its implementation. We employ rigorous analysis tools and techniques for measuring and evaluating parallel overheads and the quality of the parallelization. The outcome of these studies is that the performance of a parallel symbolic state-space generation algorithm is almost impossible to predict and highly dependent on the model to which it is applied. In most situations, perceivable speed-ups are hard to achieve, but real-world applications where our technique produces significant improvements do exist. Nevertheless, it appears that time is better invested in optimizing sequential symbolic model checking algorithms rather than parallelizing them.

DOI: http://dx.doi.org/10.1093/logcom/exp006

6.18 A PARALLEL SATURATION ALGORITHM ON SHARED MEMORY ARCHITECTURES [21]

Due to time constraints, I merely copy the abstract here:

Symbolic state-space generators are notoriously hard to parallelize. However, the Saturation algorithm implemented in the SMART verification tool differs from other sequential symbolic state-space generators in that it exploits the locality of firing events in asynchronous system models. This paper explores whether event locality can be utilized to efficiently parallelize

Saturation on shared-memory architectures. Conceptually, we propose to parallelize the firing of events within a decision diagram node, which is technically realized via a thread pool. We discuss the challenges involved in our parallel design and conduct experimental studies on its prototypical implementation. On a dual-processor dualcore PC, our studies show speed-ups for several example models, e.g., of up to 50 % for a Kanban model, when compared to running our algorithm only on a single core.

link: http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.114.1039

6.19 Can Saturation be Parallelised? [18]

Due to time constraints, I merely copy the abstract here:

Symbolic state-space generators are notoriously hard to parallelise. However, the Saturation algorithm implemented in the SMART verification tool differs from other sequential symbolic state-space generators in that it exploits the locality of firing events in asynchronous system models. This paper explores whether event locality can be utilised to efficiently parallelise Saturation on shared-memory architectures. Conceptually, we propose to parallelise the firing of events within a decision diagram node, which is technically realised via a thread pool. We discuss the challenges involved in our parallel design and conduct experimental studies on its prototypical implementation. On a dual-processor dual-core PC, our studies show speed-ups for several example models, e.g., of up to 50% for a Kanban model, when compared to running our algorithm only on a single core.

DOI: http://dx.doi.org/10.1007/978-3-540-70952-7_23

6.20 The Fortress Language Specification [1]

The Fortress programming language aims to provide ways to do multithreaded and parallel programming while still retaining some of the look of traditional programming languages and accounting for the lessons learned from the last few decades. Fortress provides a relatively traditional data type system, perhaps describable as a cross between Java types and functional types, allowing inheritance between interfaces (traits ala Java interfaces), but without traditional Class types. Most control structures in Fortress are parallel by default, with some allowing the keyword "sequential" to specify sequential behavior. Several syntactic improvements enable the use of relatively math-like notation where appropriate, and include additional opportunities for default parallelism. In particular, Set, Array, and Map comprehensions involve implicit multithreading. The syntactic enhancements also include additional mathematical operator syntax, but come at some cost, in that many rules are needed to understand exactly how some expressions will be parsed.

A hierarchical system of regions is provided to allow program control of placement of threads, and possibly data structures, although it is not clear how to control the distribution of an array. (Possibly nested) atomic statements are provided so the programmer can avoid race conditions involving shared variables.

Static parameters may be used with functions and Object types, similarly to template parameters in C++.

Contracts allow specification of preconditions, postconditions, and invariants. These must be executable, however.

Overall, ignoring certain enhancements, Fortress is about what I would expect from an effort to extend a 'normal' language, such as Java, to include support for multithreaded series-parallel computation for

'normal' processor systems, such as clusters of x86 multicores. It is an improvement over some previous attempts at parallel languages, and may remain applicable for some time. I can however easily see the possibility of a much better language being implemented as part of the current work.

Of particular interest to me is their specification of Symmetric Multiple Dispatch, the implementation of which is described in a separate paper. This relates to the repair of the Ephemeral language, which may require some sort of multiple dispatch. It is remotely possible that Fortress will make a good target language for an Ephemeral compiler.

link: http://labs.oracle.com/projects/plrg/fortress.pdf

6.21 The Scala Language Specification Version 2.9 [39]

Scala is a Java-like language with numerous extensions, supporting the development of "domain-specific" languages as libraries within Scala. To this end, parametric (with type parameters) and ad-hoc polymorphic object oriented programming is carefully supported in the language. For example, looping control structures are "virtualized", so that those control structures (say "for" loops) are syntactic sugar for something like iterator calls to the loop variable, so the meaning of loops can be changed by the type of the loop variable. Also, "Implicits" allow some call parameters to be automatically inserted by the compiler, increasing the power of user-defined libraries. An "implicit" parameter, when not given at a call site will be automatically be set to an "implicit" value of the appropriate type if exactly one is visible in the call scope. The actual rules for implicits are more complex, and also allow implicit values to be automatically used as type conversion functions, called "views". The standard library also has some predefined implicits, that allow a user-defined (polymorphic) library to obtain a "manifest" structure describing the type of data as seen by the library user.

There is considerable syntactic flexibility in definition of operator names, allowing library authors to enhance the appearance of code that uses their libraries. Scala also uses the Unicode character set, with its extra operator characters, and allows XML embedding.

The standard implementation compiles to JVM byte codes. That feature made me initially doubt the relevance of this language, since I don't consider the JVM to be a likely platform for massively parallel processing. In fact, there is no mention of parallelism or even threading in the language or the standard library specification, although all the Java libraries are fully accessible.

However, Scala seems to be near the cutting edge of research on hosting application-specific mini languages, which is an alternative approach to some goals of the current proposal. Scala library authors may effectively define (using OOP and meta-programming-like techniques) mini-languages with which to write application code.

Whereas I propose to allow user-defined code transformations in the optimization process for a declarative language, Scala allows users to embed (and optimize in any manner they wish) application-specific languages, allowing the library author and user to guarantee good performance.

The primary weakness I see with this approach, is that Scala does not enforce correctness of such libraries, as they are user-defined and specified, and the language does not provide for their formal specification. Thus, a functional error in the object code could be due to either the user or to the library author. The current proposal requires that code transformations refine entailment, so that functional errors in object code are always traceable to the application specification.

In the arena of dynamically typed languages, it is useful to remember that lisp has long been used to host embedded mini-languages, through the use of the macro mechanism.

link: http://www.scala-lang.org/node/198

6.22 Scala-Virtualized [35]

Scala-Virtualized is an extension, of the Scala language, that provides improved facilities for hosting domain-specific languages. The three major extensions are as follows: 1. "infix methods" improve syntactic flexibility by effectively allowing the addition of class members externally to the definition of the class. 2. "Fully virtualized" program structures. All program constructs are considered syntactic sugared versions of method calls, rather than just looping constructs as with Scala originally. Consequently, the object system can convert what appears to be a plain Scala program into a strongly typed abstract syntax tree which can then be processed by a user-defined library to produce transformed code. 3. Additional information about source code files is made available to library code, so that user-defined libraries may produce more meaningful error messages when used incorrectly.

Scala-Virtualized has been successfully used to embed SQL queries within Scala programs in a type-safe manner, among other impressive achievements. As Scala-Virtualized is being developed by the developers of Scala, I imagine this represents the next step in the evolution of Scala. Likewise, my comments on the Scala language also apply to Scala-Virtualized. There is no strong support for formal proof of correctness, and no strong support for parallelism or even multi-threading apart from the Java libraries.

DOI: http://dx.doi.org/10.1145/2103746.2103769

6.23 Leveraging Data-Structure Semantics for Efficient Algorithmic Parallelism [15]

This paper describes a novel programmer-assisted method for parallelizing code, based on the understanding that efficient methods for determining some program properties necessary for efficient parallelism cannot be anticipated by the compiler writer. In particular, the data footprint of a computation or of a sub-computation cannot always be represented efficiently as list of memory locations. Also, the footprint cannot always be determined statically, due to the data-dependent nature of many computations, as well as the use of pointers and indexing. The data footprint of computations can be useful for determining which computations (from existing sequentially written code) can be run in parallel. In particular, operations with non-overlapping memory footprint may be executed in parallel. Unfortunately, comparing footprints based on lists of memory locations is infeasible for parallelizing significant programs.

The described system provides C++ templates and run-time components that allow a programmer to provide additional information to enable effective parallelization. The system represents the memory footprint of an operation abstractly, using types supplied by the programmer. The programmer also provides notifications of changes in the memory footprint of an operation, as well as a way of checking overlap (and also checking probability of overlap) in footprints represented by the programmer-supplied types. At run-time, the system calls the programmer-supplied checking functions to determine suitability of allowing parallel execution of the available operations. The system does this in 2 ways. In systems with software transactional memory (STM), this information is used to throttle concurrency based on probability of rollback. For other systems, it is used to obtain guarantees of non-interference before allowing parallel execution.

This can work well, because the programmer often knows an efficient representation of the desired footprint. For example, operations that each perform incremental operations on trees, then recursively descending on subtrees, have a footprint that can be conservatively be represented by reference to the node on which they are currently operating, the footprint being the subtree under that node. In this case, overlap between two footprints can be checked by inspecting the relation between the two referenced nodes.

The disadvantage of this approach (aside from the obvious fact that this implementation only applies to C++) is that there are no correctness guarantees as there are with some systems that rely on static

analysis.

This is a very good example of a system that utilizes programmer knowledge (expressed procedurally, in this case) that potentially goes far beyond what can be anticipated by the toolmaker (or compiler writer), to enable greater program performance improvements.

DOI: http://dx.doi.org/10.1145/2016604.2016638

6.24 Distillation with Labelled Transition Systems [27]

The "Distillation" transform for functional programs was introduced in 2007 as an automatic way to perform certain transforms of list processing code previously thought to require mathematical insight. This paper discusses an explanation of how the Distillation transform sometimes produces a genuine algorithmic improvement having super-linear speedup. A correctness proof is also sketched.

This is vaguely reminiscent of a 1984 Lisp conference paper subtitled "Listlessness is better than laziness".

This paper provides yet another example of the many optimization techniques that are somewhat specialized and hence not desirable to incorporate into a general purpose compiler, yet are highly desirable for many programs, necessitating its use in some compilers.

The proposed work solves this dilemma by providing a framework in which a user may safely define and invoke this optimization without modifying the compilation system itself.

DOI: http://dx.doi.org/10.1145/2103746.2103753

6.25 StagedSAC: A Case Study in Performance-Oriented DSL Development. [50]

This paper describes two implementations of SAC (Single Assignment C)-like languages in the Scala-Virtualized framework. SAC is a C-like language with a single-assignment rule for variables, and some additional array manipulation constructs intended for parallel implementation. In the first implementation, "LibrarySAC", a library is defined, utilizing the syntactic flexibility of Scala to provide the programmer with a way of writing SAC-like code within Scala. SAC functionality is then available within Scala through the use of library calls.

The second implementation, "StagedSAC", is accessed through library calls within Scala-Virtualized, but achieves higher performance through code optimizations and a "lightweight modular staging" (LMS) framework, also implemented in Scala-Virtualized. The full virtualization of Scala-Virtualized allows the library code to extract abstract syntax trees of the StagedSAC portion of the code. The library, at compile time, then performs transformations and optimizations on the code, resulting in improved Scala code. A constraint system on array shapes is derived from the program graph, the solution of which sometimes provides partial information on array shapes. Whatever partial information was derivable at compile time about array shapes is then used to optimize array code, via such improvements as removal of redundant index bounds checks, and later, loop specialization. All this information is passed to the LMS framework, which provides common optimizations such as constant folding, CSE, code motion, etc. Other optimizations such as tiling may be applied to improve cache effectiveness. Upon request, the Delite framework may also be used to do additional optimizations and generation of GPU code. One of the major contributions of this work is the ability to use partial shape information in certain ways for optimizing array code.

A results section provides comparisons between various levels of optimization of the StagedSAC implementation (executing on JVM) and a native SAC implementation, for some sample array programs (not

using GPU). The fully implemented StagedSAC programs running on JVM had run-times within an order of magnitude of that of their corresponding natively compiled SAC programs.

This illustrates Scala being used for what it does best, the implementation and customized optimization of "Domain-Specific" languages within the bounds of interoperability within a general purpose strongly typed language.

DOI: http://dx.doi.org/10.1145/2103746.2103762

6.26 Active Pebbles: Parallel Programming for Data-Driven Applications [53]

Active Pebbles attempts to provide a framework for message-passing parallel programs that naturally utilize small messages with little or no coherence or predictability in the message flow patterns. Such programs may experience performance problems on traditional platforms, due to their high per-message overhead for inter-processor communications. Pebbles are (potentially very small) messages sent between entities. There may be very many senders, and very many receivers, called targets (also potentially very small). One may imagine the data flow pattern as a storm in a room full of tiny pebbles. There is no expectation of regular patterns or coherence. They provide results showing that Active Pebbles gives good performance for various benchmarks on top of various parallel programming platforms. The authors list 5 mechanisms responsible for the performance of their framework.

1 Fine-Grained Pebble Addressing: Pebbles are addressed directly to their target, with an address of reasonable size.

2 Message Coalescing: Pebbles may be aggregated temporarily into larger messages by the AP mechanisms, to decrease the messaging overhead on platforms that only efficiently support larger messages.

3 Active Routing: Pebble flow is slightly adjusted, to increase the probability of message coalescing.

4 Message Reduction: Depending on program semantics, certain pebbles may be reduced in transit, if they are coalesced into the same message.

5 Termination Detection: Support is provided for some distributed quiescence detection algorithms.

I find 1, 2, and 3 most interesting, as these are the mechanisms that overcome the small-message penalty on some platforms. This occurs with the penalty of small additional latencies introduced by the active routing and coalescence mechanisms. Fortunately, these mechanisms are programmable, and can be adjusted to also efficiently handle workloads with well-understood communication patterns. The observed performance justifies my assumption (in Ephemeral) that small messages (within the computing/programming model) are reasonably efficient and sufficient for all parallel programming needs.

This may provide a convenient implementation target for the parallel compiler, if there is not sufficient time to implement Ephemeral. If there is time to implement Ephemeral, it would be wise to consider use of some mechanisms from Active pebbles.

DOI: http://dx.doi.org/10.1145/1995896.1995934

6.27 Position paper: Using a "Codelet" Program Execution Model for Exascale Machines. [56]

The authors observe that course-grained parallelism, of the type that is efficiently supported by extant architectures, denies the run-time system of certain opportunities for adaptation, and tends to require large

overheads for certain operations, such as task swapping and migration. The authors claim that their work indicates improvements are possible by dividing the program into smaller pieces (codelets) for execution.

The Codelet model they propose appears to be a hybrid between Ephemeral and the dataflow model.

It is difficult for me to see this work as original, as I have seen many hybrid dataflow systems proposed since the early 1980's, and the dataflow model is certainly no stranger to the use of small executable units. The part that seems novel to me is the claim that their work supports this. It's too bad the paper gives no details about, or references to such work. But I suppose that is to be expected from a paper that mentions IBM Cyclops-64, and Intel's single chip cloud machine, but not Sun's Niagara.

A generic hardware model is also mentioned, having a hierarchy of 4 levels (system, node, chip, and cluster), the typical unit for each level containing an interconnect and multiple units of a lower level attached to the interconnect. The node level unit also has extra DRAM banks attached to its interconnect. The lowest, cluster, level has computing units (CU) and at least one scheduling unit (SU), and a cluster memory, and interconnect between them. CUs and SUs have local memory and multiple register sets. SUs also have the ability to communicate with SUs in other clusters, presumably for load balancing and migration.

The relation between the hardware model and the codelet model is not clearly explained. My guess is that this paper was trimmed from a larger paper, and the hardware model was left in to provide some notion of what kind of computer would benefit from use of the codelet model.

DOI: http://dx.doi.org/10.1145/2000417.2000424

6.28 Adaptive Runtime Selection of Parallel Schedules in the Polytope Model [42]

This is limited to Polytope model computations, but has improved accuracy compared to many other techniques. The limitation to possibly parametric polytope model computations allows analytical determination of loop iteration counts and array sizes after the input array sizes are known. This, in turn, allows improved performance prediction.

The compiler may produce multiple versions of the code depending on how flexible the array data dependencies are. Their method produces a modified code at compile time which performs profiling on itself, along with the production code for performance execution. At install time, each code version is profiled systematically, with various numbers of threads, various input array sizes/shapes, and various tile sizes. The profile data, after tabulation, effectively defines a performance model for the code. Finally, at run time, the input sizes are known, and the number of available processors/threads is known approximately, so that the model can be used to predict the performance of the best variation of each version of the code. For each version, the best parameters can be predicted, such as number of threads to use, and tile sizes, depending on input sizing and processor loading, etc.

This method performs quite well at selecting a good version of the code to execute, because it is able to account for almost all factors influencing performance. These include: processor design (considered by load-time profiling), Input data size/shape (accounted for by systematic profiling using Polytope model) Processor resource availability/loading (via profiling with various numbers of threads)

What I find relevant about this work is that it explores one of many instances where a large class of computational problems (HPC codes) which are in general difficult to optimize, has a frequently occurring subset of problems (in this case, problems expressible as coherent loop nests with simple dependencies) which, as this research now shows, are relatively easy to optimize well, even for parallel processing.

My proposed project will provide a framework in which such cases may be defined and recognized, and in which to express what needs to be done in such cases, without requiring actual compiler modification.

link: http://dl.acm.org/citation.cfm?id=2048588

6.29 The Elephant and the Mice: The Role of Non-Strict Fine-Grained Synchronization for Modern Many-Core Architectures [43]

This paper explores 4 questions, among them is the performance gain achievable by the use of non-strict fine-grained synchronization (dataflow tokens, full/empty tag bits, synchronization state buffer), as compared with other synchronization mechanisms (barriers, signal-wait).

The authors implemented three fine-grained synchronization mechanisms for the (single-chip) IBM Cyclops-64, and tested it in very carefully crafted simulations to ensure accuracy of results. The IBM Cyclops-64 has a relatively symmetric NUMA architecture with a large crossbar and 160 "thread units" (single issue cores) in pairs having a shared FPU and a crossbar port. Each group of 10 thread units also share an instruction cache, every four of which also share a crossbar port. Each core has its own data memory, and direct (but higher latency) access to all other cores data memories through the crossbar. There are no data caches. The cores have scoreboarding which allows some out-of-order execution and write back. There is also a common high-speed synchronization bus shared by all cores.

The mechanism(s) implemented were integrated deeply into the architecture in the form of an Extended Synchronization State Buffer (ESSB). The ESSB essentially simulates the use of imaginary tag bits attached to some memory locations chosen implicitly by the program. Special load and store instructions utilize the ESSB. In the most effective mechanism tried, ESSB3, a special store sets the full bit associated with the memory location, after which the special load instruction resets it. If the load instruction occurs before the needed value has been stored to the location, a stall may eventually occur, but the thread will be automatically resumed once the data becomes available. The load instruction still issues but the thread will not stall until the value itself is needed by an operation.

The ESSB mechanisms, along with barriers and signal-wait, were all tested with customized versions of a number of benchmarks. As hoped for, the ESSB3 versions of all programs scaled well as long as there was available parallelism, while performance of the equivalent versions using other mechanisms became limited due to synchronization overhead. In particular, a case of the wavefront benchmark gave almost linear speed up out to 160 threads with ESSB3, while the best of the other methods (signal-wait) only scaled to about 115 threads.

One of the other results of the paper is that introduction of fine-grained synchronization increases the hardware size by at most 10%, that almost entirely due to the ESSB cache-like structure itself.

I claim this provides additional justification for doing things in a fine-grained manner in parallel computing models, as in Ephemeral, although this paper only addresses synchronization methods.

DOI: http://dx.doi.org/10.1145/1995896.1995948

6.30 Programmable Data dependencies and Placements [7]

This paper is based on the factoring of a (data-independent) program into a data dependency graph, and the individual computations performed at graph nodes. The proposed use of a data dependency algebra (DDA) to generate the dependency graph is described, as well as use of a space-time DDA (STA) to describe the communication topology of a parallel processor architecture. The main idea is that a (data-independent) program can be expessed as the following three modules:

1. The algorithm, independent of data dependency.
2. The data dependency graph expressed as a DDA.
3. The mapping/embedding from the DDA to a the STA of the execution platform.

and that 3 can be generated automatically from 2 and the platform STA, while the programmer factors the program into 1 and 2. The compiler writers are to be responsible for generating the STA graph descriptions of target platforms. The paper goes on to describe DDAs for various parallel algorithms, then STAs for various processing topologies, then various embeddings of DDAs into STAs, and then code generation. The system was not yet implemented, so the performance results are due to manually "compiled" code.

This system is simillar to a proprietary system "GAUSS" that I 'almost' implemented in 1989. GAUSS did not require programs to be data-independent, however data-dependent programs required more programmer annotation (manual allocation of processor resources).

It is also simillar to another proprietary system I proposed later in 1989, that would automatically handle C code, limited to the data-independent case.

This paper, together with some of its referrences, shows that automatic placement of computations onto processors in various topologies has long advanced to a point sufficient to support many practical high performance applications.

DOI: http://dx.doi.org/10.1145/2103736.2103741

6.31 Expressive array constructs in an embedded GPU kernel programming language [14]

This paper describes the experimental addition of a secondary kind of array to the Obsidian GPU Kernel Programming Language.

Obsidian is a Haskell package for generating GPU kernels. Using Obsidian, one writes Haskell code that produces a Kernel. Unlike many code generators, the Obsidian/Haskell program resembles the generated kernel, so the programming process is more like writing a kernel in a high level language than like writing a code generator. Obsidian provides GPU-limited versions of the usual programming constructs, including arrays. The existing array construct, "pull" arrays, may be read using complex index expressions, but may only be constructed (written) using coherent indexing, such as an affine function of the threadid. This works quite well sometimes, for example, allowing a simple form of automatic loop fusion. In some cases, such as array concatenation, this is inefficient, as it requires use of conditionals within a kernel. The situation is improved through the use of the novel "push" arrays. Push arrays may be written using complex indexing expressions, but may not be read that way. After construction, a push array may then be effectively converted to a pull array, requiring insertion of synchronization code between these different uses. The paper proceeds to illustrate the use of push arrays via simple parallel sorting kernels (Batcher's bitonic algorithm) and performance measurements.

This is another example of a case where a particular optimization (separation of gather and scatter operations into layers with intervening synchronization), has the following characteristics: One would not expect it to be built into any compiler, but research shows it is quite effective for certain specialized situations. It seems obvious enough to be discoverable by any serious programmer.

DOI: http://dx.doi.org/10.1145/2103736.2103740

6.32 The Sequential Prison [49]

The computer graphics (and OOP) pioneer, Ivan Sutherland, makes the case that the self-propagating cycle of learning and teaching sequential programming has become so entrenched that there will be no escape without a significant change in the way computation is seen. More specifically, he claims that even the use of languages encoded as sequences of characters causes a sequential bias in our thinking about computations so expressed. Many other observations are made in his talk. Asynchronous logic is almost never used in modern systems, even though it potentially offers considerable energy savings. This may

be evidence that computer engineers are stuck in a rut (sequential clocked vs. self-timed logic) similar to that (sequential vs. parallel) of computer scientists. The way programming is described contributes to the sequential prison. A program is usually defined as a sequence of steps. Sutherland says that computer scientists need to stop using the word "programming" for what they do, and use another word, such as (perhaps) "configuration", if they are to ever escape the rut.

The current proposal seeks to avoid becoming trapped in the sequential prison, in various ways including the following: Predicate logic programming is used at all levels, so that effort is required to introduce sequencing. Only an abstract syntax is defined, to avoid the necessity of expressing programs as a sequence of bytes. The system target is the Ephemeral language, which itself avoids sequential bias.

DOI: http://dx.doi.org/10.1145/2048066.2048068
VIDEO: http://dl.acm.org/ft_gateway.cfm?id=2048068&ftid=1163270
&dwn=1&CFID=120492121&CFTOKEN=31776533

6.33 Adapt or become extinct! The Case for a Unified Framework for Deployment-Time Optimization [24]

I should perhaps take this advice, as this project was hatched in the early 80s. =)

This paper advocates the adoption of adaptation in many forms as a necessity for future high - performance computing applications, especially for scaling the "walls" of memory, communications, parallel programming, power, etc.

It appears that homogeneity may never arrive in the area of high-performance/high-efficiency computing the way it has for desktop computing. This in mind, the authors point out that a code that is tuned for one system is quite likely to perform poorly on another system, due to potential differences in a variety of system attributes, all of which may need to be accounted for in the code if it is to perform well. These attributes include ISA, number of processors, memory per processor, interconnection network, cache sharing and cache hierarchy, among others.

Various extant adaptation strategies are discussed, such as optimizing compilers, algorithms with various adjustable performance parameters, auto-tuning libraries, choice of different communication packages, scheduling methods, and cache-coherence protocols.

It is also pointed out that sometimes adaptation must be done at run-time, for example choosing matrix representations and algorithms based on the sparseness of data and on the nature of sub-structures. Another example is the use of schedulers that schedule complementary workloads together to optimize resource utilization. Another example is use of profiling data.

It is also pointed out that all these techniques are applied in a rather ad-hoc manner, and that a more holistic approach will be necessary in the future. The authors propose an "adaptation infrastructure" where a single decision maker receives information both at run-time, and before, in the form of programmer annotations, static analysis, profile data etc., and uses all this information to make adaptation decisions of all kinds, both before and during run-time.

I would only point out that having a programming language capable of expressing and controlling compiler optimizations could greatly simplify the task of creating such an infrastructure, especially in the case where correctness is required.

DOI: http://dx.doi.org/10.1145/2000417.2000422

6.34 Implementation of a Hierarchical N-Body Simulator Using The OmpSs Programming Model [41]

This paper describes lessons learned parallelizing Treecode, an N-body gravitation simulator using the Barns-Hut algorithm, for execution on a moderately (4 6-core Xeon with a total of 48 GB) parallel system.

The OmpSs system is an extension of OpenMP, with additional annotations to designate data flow directionality between tasks. OmpSs also adds an extra thread for each processor to manage data-flow-like task synchronization.

The authors consider this algorithm to be a representative example of "irregular scientific applications", in that it solves a problem that could be solved using array computing on a regular grid, but requires less computation, by doing things in a data-dependent manner.

Even on this system with large traditional processors having large memories, the main lesson reported is that finer grained tasks were much better for load balancing, and that processors should include support for task creation and scheduling for larger numbers of smaller tasks to improve processor utilization.

DOI: http://dx.doi.org/10.1145/2089142.2089150

6.35 Communication and Concurrency [33]

This is Milner's famous reference on verification of concurrent systems, which contains the definition of bisimulation I use. It also contains the 'round-robin' scheduler model used in some of my benchmarks in bisimulation research.

6.36 Concurrency and Automata on Infinite Sequences [40]

This is another famous reference on bisimulation, which must be referenced, but which I did not actually read. I copy the abstract here:

> The paper is concerned with ways in which fair concurrency can be modelled using notations for omega-regular languages languages containing infinite sequences, whose recognizers are modified forms of Büchi or Muller-McNaughton automata. There are characterization of these languages in terms of recursion equation sets which involve both minimal and maximal fixpoint operators. The class of ω-regular languages is closed under a fair concurrency operator. A general method for proving/deciding equivalences between such languages is obtained, derived from Milner's notion of simulation.

DOI: http://dx.doi.org/10.1007/BFb0017309

6.37 Graph-Based Algorithms for Boolean Function Manipulation [6]

Due to time constraints, I merely copy the abstract here:

> In this paper we present a new data structure for representing Boolean functions and an associated set of manipulation algorithms. Functions are represented by directed, acyclic graphs in a manner similar to the representations introduced by Lee [1] and Akers [2], but with further restrictions on the ordering of decision variables in the graph. Although a function requires, in the worst case, a graph of size exponential in the number of arguments, many of the functions encountered in typical applications have a more reasonable representation. Our algorithms have time complexity proportional to the sizes of the graphs being operated on, and hence are quite efficient as long as the graphs do not grow too large. We present experimental results

from applying these algorithms to problems in logic design verification that demonstrate the practicality of our approach.

DOI: http://dx.doi.org/10.1109/TC.1986.1676819

6.38 MACLISP Reference Manual [34]

This manual contains all that is needed to understand the detailed operation of MACLISP. It was from this manual that I first learned the operation of LISP macros. As LISP operates by eager evaluation, macros provide one of the few ways to simulate lazy evaluation. The evaluator will call the macro (possibly at run-time) using the source code of the un-evaluated arguments, so that the macro may give to the arguments whatever semantics are desired. Proper use of the argument source code by the macro requires care, as evaluating such source may accidentally cause references to the internal variables of the macro, instead of the entity they statically appear to reference. This and other related problems eventually lead to the invention of 'Hygienic' Macro Expansion, discussed next.

6.39 Hygienic Macro Expansion [30]

I viewed the presentation of this paper at the 1986 conference on LISP and Functional Programming. The authors presented a workable solution to the problems of incorrect references that can occur when using macros in non-statically scoped variants of LISP. It was noted that unfortunately, the solution was implemented entirely in the interpreter and other such supporting code, in such a way that the solution was not clearly describable within the language itself, so that a meta-level solution could not be constructed (within user-modified evals presumably) within the language as implemented. Thus, there remained some messiness clouding the meaning of macro expansions.

DOI: http://dx.doi.org/10.1145/319838.319859

6.40 Syntactic Closures [2]

This paper (which I also viewed at the 1988 conference on LISP and Functional Programming), Presented a solution to the messiness problem for the 'extend-syntax' feature of the R^3 Scheme programming language. Again, however, there was unfinished business. It appeared that the language implementation had to manipulate things in a way that programs in the language could not, in order to make things work cleanly.

DOI: http://dx.doi.org/10.1145/62678.62687

6.41 A Confluent Calculus of Macro Expansion and Evaluation [5]

Due to time constraints, I merely copy the abstract here:

> Syntactic abbreviations or macros provide a powerful tool to increase the syntactic expressiveness of programming languages. The expansion of these abbreviations can be modeled with substitutions. This paper presents an operational semantics of macro expansions and evaluation where substitutions are handled explicitly. The semantics is defined in terms of a confluent, simple, and intuitive set of rewriting rules. The resulting semantics is also a basis for developing correct implementations.

DOI: http://dx.doi.org/10.1145/141471.141562

6.42 The C++ Programming Language [47]

Due to time constraints, I merely copy the abstract here:

> Written by Bjarne Stroustrup, the creator of C, this is the world's most trusted and widely read book on C. For this special hardcover edition, two new appendixes on locales and standard library exception safety have been added. The result is complete, authoritative coverage of the C language, its standard library, and key design techniques. Based on the ANSI/ISO C standard, The C Programming Language provides current and comprehensive coverage of all C language features and standard library components. For example: abstract classes as interfaces class hierarchies for object-oriented programming templates as the basis for type-safe generic software exceptions for regular error handling namespaces for modularity in large-scale software run-time type identification for loosely coupled systems the C subset of C for C compatibility and system-level work standard containers and algorithms standard strings, I/O streams, and numerics C compatibility, internationalization, and exception safety Bjarne Stroustrup makes C even more accessible to those new to the language, while adding advanced information and techniques that even expert C programmers will find invaluable.

6.43 The C++ Programming Language [48]

Due to time constraints, I merely copy the abstract here:

> C++11 has arrived: thoroughly master it, with the definitive new guide from C++ creator Bjarne Stroustrup, C++ Programming Language, Fourth Edition! The brand-new edition of the world's most trusted and widely read guide to C++, it has been comprehensively updated for the long-awaited C++11 standard. Extensively rewritten to present the C++11 language, standard library, and key design techniques as an integrated whole, Stroustrup thoroughly addresses changes that make C++11 feel like a whole new language, offering definitive guidance for leveraging its improvements in performance, reliability, and clarity. C++ programmers around the world recognize Bjarne Stoustrup as the go-to expert for the absolutely authoritative and exceptionally useful information they need to write outstanding C++ programs. Now, as C++11 compilers arrive and development organizations migrate to the new standard, they know exactly where to turn once more: Stoustrup's C++ Programming Language, Fourth Edition.

6.44 Preliminary Ada Reference Manual [29]

I learned Ada from this version of the Ada Reference Manual in 1980. It is interesting to remember that generic procedures was in its own separate chapter from non-generic procedures. The use of generics in Ada is quite strongly typed, and appears to be a definite advance over the use of macros when program understandability is of paramount importance. Ada generics are, of course, much less flexible than C++ templates, but, at the time, I saw that the Ada type system was at least more flexible than that of Pascal. The Ada language remained essentially unchanged until the revisions by the Ada9X efforts, which were probably implemented sometime after 1999.

DOI: http://dx.doi.org/10.1145/956650.956651

6.45 The Java Language Specification, Java SE 7 Edition [23]

Due to time constraints, I merely copy the abstract here:

Written by the inventors of the technology, The Java Language Specification, Java SE 7 Edition, is the definitive technical reference for the Java programming language. The book provides complete, accurate, and detailed coverage of the Java programming language. It fully describes the new features added in Java SE 7, including the try-with-resources statement, multi-catch, precise rethrow, diamond syntax, strings-in-switch, and binary literals. The book also includes many explanatory notes, and carefully distinguishes the formal rules of the language from the practical behavior of compilers.

6.46 C++ Seminar [45]

Due to time constraints, I merely copy the abstract here:

> C++ usage has changed drastically over the past ten years with much more advanced use of templates, meta-programming, functors, and other high-level concepts becoming common. Last year, a new C++ standard (C++11, formerly C++0x) was released with many new and exciting changes to the language that follow these trends.

URL: `http://www.cs.ucr.edu/~cshelton/cppsem.cgi`

6.47 Impact of Economics on Compiler Optimization [44]

Due to time constraints, I merely copy the abstract here:

> Compile-time program optimizations are similar to poetry: more are written than are actually published in commercial compilers. Hard economic reality is that many interesting optimizations have too narrow an audience to justify their cost in a general-purpose compiler, and custom compilers are too expensive to write. An alternative is to allow programmers to define their own compile-time optimizations. This has already happened accidentally for C++, albeit imperfectly, in the form of template metaprogramming. This paper surveys the problems, the accidental success, and what directions future research might take to circumvent current economic limitations of monolithic compilers.

DOI: http://dx.doi.org/10.1145/376656.376751

6.48 Pure Logic Program Compilation and Improvement (Optimization) Via Programmer-Controlled Transformations [36]

Due to time constraints, I merely copy the abstract here:

> I describe a novel hybrid approach to efficient parallel implementation of declaratively (and formally) specified applications. My approach proposes novel languages that support formal refinement and parallel distributions. I defer the complexity of designing user-level programming language features in favor of designing intermediate languages with specific properties and sufficient generality.
>
> My primary high-level intermediate language (IL), RILL (Refinement-based Intermediate Logic Language), provides the generality of predicate logic for specification and supports arbitrarily powerful programmer-controlled optimizations, through formal refinement. My secondary IL, Ephemeral, supports low-level parallel processing across a wide variety of platform types. A programmer writes a formal specification in a syntactically pleasing 'surface' language,

and invokes a surface language compiler, which translates the specification into RILL, and invokes my proposed system. My proposed system starts with the formal specification translated to a RILL program, optimizes this program via correctness preserving RILL-to-RILL code refinements, then translates the optimized code to Ephemeral, invoking the Ephemeral compiler for translation to specific platforms.

My approach to compiling predicate logic programs in RILL is novel in the way it avoids the generally intractable problem of compiling declarative specifications. I take advantage of the fact that existing solutions to many computing problems can be construed as solutions of specific declarative specifications. I additionally exploit the wealth of existing decision procedures for many classes of problems, such as FOL, SAT, ILP, linear systems, etc., using them as bases for refinement methods. The most universally necessary refinement methods should occupy a 'standard library' of optimization procedures, leaving less universal and more advanced optimizations to extended libraries. RILL provides the novel *Refine* construct, which application programmers use to control the application of optimization refinements that transform their declarative specifications to a more executable and efficient form. When compositions of available refinements do not suffice to optimize a particular specification, the Refine construct also provides the means of applying custom optimizations, which may be constructed by the application programmer. Thus, the implementation of RILL is a step toward obtaining high application performance with a general purpose logic programming language.

The proposed low-level parallel programming language, Ephemeral, mostly avoids architectural dependence by making only minimal assumptions about supported platforms. Ephemeral uniquely supports MIMD arrays with tiny individual processors, not having sufficient memory for the entire program, such as I expect will eventually dominate the high end of the MIMD performance spectrum. The possibility of high application performance is retained by allowing distribution, of both data and computation, to be specified in a somewhat topology dependent manner. The geometry of Ephemeral's computational model is physics-based, ensuring performance metrics based on the model accurately bound communication performance of compiled code executing on parallel platforms.

The resulting hybrid RILL/Ephemeral system will support the declarative specification and implementation of efficient parallel applications. Thus, this research moves toward positively answering the question:

"Can we escape the Sequential Prison by programming in a general purpose declarative language?".

`URL: http://www.cs.ucr.edu/~mummem/ProposalABr1.pdf`

6.49 A Symbolic Algorithm for Optimal Markov Chain Lumping [16]

Due to time constraints, I merely copy the abstract here:

Many approaches to tackle the state explosion problem of Markov chains are based on the notion of lumpability, which allows computation of measures using the quotient Markov chain, which, in some cases, has much smaller state space than the original one. We present, for the first time, a symbolic algorithm and its implementation for the lumping of Markov chains that are represented using Multi-Terminal Binary Decision Diagrams. The algorithm is optimal, i.e., generates the smallest possible quotient Markov chain. Our experiments on various configurations of two example models show that the algorithm (1) handles significantly

larger state spaces than an explicit algorithm, (2) is in the best case, faster than an efficient explicit algorithm while not prohibitively slower in the worst case, and (3) generates quotient Markov chains that are several orders of magnitude smaller than ones generated by a model-dependent symbolic lumping algorithm.

DOI: http://dx.doi.org/10.1007/978-3-540-71209-1_13

6.50 Using Edge-Valued Decision Diagrams for Symbolic Generation of Shortest Paths [12]

Due to time constraints, I merely copy the abstract here:

We present a new method for the symbolic construction of shortest paths in reachability graphs. Our algorithm relies on a variant of edge-valued decision diagrams that supports efficient fixed-point iterations for the joint computation of both the reachable states and their distance from the initial states. Once the distance function is it known, a shortest path from an initial state to a state satisfying a given condition can be easily obtained. Using a few representative examples, we show how our algorithm is vastly superior, in terms of both memory and space, to alternative approaches that compute the same information, such as ordinary or algebraic decision diagrams.

DOI: http://dx.doi.org/10.1007/3-540-36126-X_16

6.51 SMART: Stochastic Model checking Analyzer for Reliability and Timing, User Manual [11]

Due to time constraints, I merely copy the abstract here:

SMART is a software package that integrates various high-level logical (functional) and timing/stochastic (nonfunctional) modeling formalisms (e.g., stochastic Petri nets) in a single modeling study. Each (sub)model is described in a uniform environment and solved using a variety of solution techniques, from symbolic model-checking for temporal logic verification to numerical methods and simulation for performance analysis. Since SMART is intended as a research tool, it is written in a modular way that allows researchers to perform the easy integration of new formalisms and solution algorithms. One of the main strengths of SMART is its emphasis on structural decomposition methods for the efficient storage and analysis of discrete-state models.

URL: http://www.cs.ucr.edu/~ciardo/SMART/

6.52 Symbolic State-Space Generation of Asynchronous Systems Using Extensible Decision Diagrams [51]

Due to time constraints, I merely copy the abstract here:

We propose a new type of canonical decision diagrams, which allows a more efficient symbolic state-space generation for general asynchronous systems by allowing on-the-fly extension of the possible state variable domains. After implementing both breadth-first and saturation-based state-space generation with this new data structure in our tool SmArT, we are able to

exhibit substantial efficiency improvements with respect to traditional static decision diagrams. Since our previous works demonstrated that saturation outperforms breadth-first approaches, saturation with this new structure is now arguably the state-of-the-art algorithm for symbolic state-space generation of asynchronous systems.

DOI: http://dx.doi.org/10.1007/978-3-540-95891-8_52

6.53 Symbolic computation of strongly connected components and fair cycles using saturation [55]

Due to time constraints, I merely copy the abstract here:

The computation of strongly connected components (SCCs) in discrete-state models is a critical step in formal verification of LTL and fair CTL properties, but the potentially huge number of reachable states and SCCs constitutes a formidable challenge. We consider the problem of computing the set of states in SCCs or terminal SCCs in an asynchronous system. We employ the idea of saturation, which has shown clear advantages in symbolic state-space exploration (Ciardo et al. in Softw Tools Technol Transf 8(1):425, 2006; Zhao and Ciardo in Proceedings of 7th international symposium on automated technology for verification and analysis, pp 368381, 2009), to improve two previously proposed approaches. We use saturation to speed up state exploration when computing each SCC in the Xie-Beerel algorithm, and we compute the transitive closure of the transition relation using a novel algorithm based on saturation. Furthermore, we show that the techniques we developed are also applicable to the computation of fair cycles. Experimental results indicate that the improved algorithms using saturation achieve a substantial speedup over previous BFS algorithms. In particular, with the new transitive closure computation algorithm, up to 10150 SCCs can be explored within a few seconds.

DOI: http://dx.doi.org/10.1007/s11334-011-0146-3

6.54 Forwarding, Splitting, and Block Ordering to Optimize BDD-based Bisimulation Computation [54]

Due to time constraints, I merely copy the abstract here:

In this paper we present optimizations for a BDD-based algorithm for the computation of several types of bisimulations which play an important role for minimisation of large systems thus enabling their verification. The basic principle of the algorithm is partition refinement. Our proposed optimizations take this refinement-structure as well as the usage of BDDs for the representation of the system into account: (1) *block forwarding* updates in-situ newly refined blocks of the partition, (2) *split-driven refinement* approximates the blocks that may be refined, and (3) *block ordering* heuristically suggests a good order in which the blocks will be refined.

We provide substantial experimental results on examples from different applications and compare them to alternative approaches. The experiments clearly show that the proposed optimization techniques result in a significant performance speed-up compared to the basic algorithm as well as to alternative approaches.

link: http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.65.655

6.55 Symbolic bisimulation minimisation [4]

Due to time constraints, I merely copy the abstract here:

> We describe a set of algorithmic methods, based on symbolic representation of state space, for minimisation of networks of parallel processes according to bisimulation equivalence. We compute this with the Coarsest Partition Refinement algorithm, using the Binary Decision Diagram structures. The method applies to labelled synchronised vectors of finite automata as the description of systems. We report performances on a couple of examples of a tool being implemented.

DOI: http://dx.doi.org/10.1007/3-540-56496-9_9

6.56 Approximate steady-state analysis of large Markov models based on the structure of their decision diagram encoding [52]

Due to time constraints, I merely copy the abstract here:

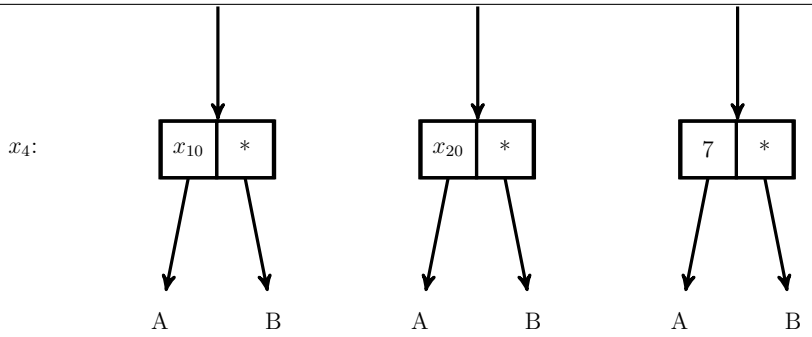
> We propose a new approximate numerical algorithm for the steady-state solution of general structured ergodic Markov models. The approximation uses a state-space encoding based on multiway decision diagrams and a transition rate encoding based on a new class of edge-valued decision diagrams. The new method retains the favorable properties of a previously proposed Kronecker-based approximation, while eliminating the need for a Kronecker-consistent model decomposition. Removing this restriction allows for a greater utilization of event locality, which facilitates the generation of both the state-space and the transition rate matrix, thus extends the applicability of this algorithm to larger and more complex models.

DOI: http://dx.doi.org/10.1016/j.peva.2011.02.005

URL: http://www.cs.iastate.edu/~ciardo/pubs/2011PEVA-Approx.pdf

7 Acknowledgements

I acknowledge the Divine Providence of my Lord Jesus, the Christ, especially in showing that the saturation heuristic would work with non-deterministic bisimulation, and lumping in 2009, and for making obvious the canonicity of GDDs. I also thank Him for bringing me to the right advisor at the right time. I thank my advisor for supporting my travel to RP2011, and for suggesting the lumping problem in 2008.



(a.) Suppose $x_{10} = 7$ (b.) Suppose $x_{20} = 7$ (c.)

Fig. 9: These GDDs at level 4 behave identically when $x_{10} = x_{20} = 7$ holds in the prefix.

A Mathematics and proofs related to contributions

A.1 Unambiguous GDDs are self-consistent

Here I show that for any unambiguous GDD A, with child B, the recursive decoding process for f_A never uses f_B if the argument x does not satisfy C_B. The demonstration is in two parts:

(Part 1) Assuming $k = level(A) \wedge x_{\uparrow k} \in sat(A) \wedge A[r] = B$, for some edge label $r \in labels(A)$, we must show $x_{\uparrow k}x_k \in sat(B)$ where $x_k = eval_{x_{\uparrow k}}(r)$ so Equation 5 applies when evaluating f_A.

Every constraint $c_1 \in C_B$ is of the form $x_i \neq c$ or of the form $x_i \neq x_j$, so we know $i \neq k$ (or $i \neq k \wedge j \neq k$) because references to x_k cannot occur within B due to rule 6. Rule 4 also applies, so that we know the constraint c_1 is also in C_A. Consequently $x_{\uparrow k} \in sat(B)$ because $x_{\uparrow k} \in sat(A)$, and $x_{\uparrow k}x_k \in sat(B)$, where $x_k = eval_{x_{\uparrow k}}(r)$, because x_k is not constrained by $sat(B)$.

(Part 2) We must also show $x_{\uparrow k}x_k \in sat(B)$ whenever $\forall p \in labels(A) : x_k \neq eval_{x_{\uparrow k}}(p)$ so Equation 7 is used to evaluate f_A, assuming $k = level(A) \wedge x_{\uparrow k} \in sat(A) \wedge A[\text{`*'}] = B$.

For any constraint in C_B which does not reference x_k, the logic in part 1 still applies, so all that remains is to consider constraints $c_1 \in C_B$ of the form $x_k \neq q$ where label $q \neq$ '*'. In these cases, rule 5 applies, so we know that $A\overrightarrow{q}$. For any tuple x that violates c_1, we have $x_k = eval_{x_{\uparrow k}}(q)$. In that case, the evaluation of f_A depends on Equation 5 rather than on Equation 7, so label q will be followed instead of label '*', hence f_B will not be used.

A.2 Equivalence for GDDs

As $f_A(x)$ is only well defined for tuples x where $x_{\uparrow level(A)} \in sat(A)$, I define functional equivalence of GDDs (A and B) as follows:

$\forall A, B \in GDD_k^K, k < K :$

$A \underset{f}{=} B$ iff: $\forall x \in \mathbb{N}^K \cap sat(A) \cap sat(B) : f_A(x) = f_B(x)$.

However, for purposes of proving canonicity, we would like to prove functional non-equivalence of non-identical GDDs. Certain prefixes, however, make some non-identical GDDs behave equivalently, as shown in Fig. 9.

For my proof, I will use a form of equivalence that avoids the problematic prefixes.

The problem arises when a variable in the prefix corresponding to a variable name in one of the GDDs has the same value as a constant in one of the GDDs, or when two variables in the prefix corresponding to variable names in one of the GDDs have the same value.

I avoid this problem by focusing on tuples where certain ('free') variables are chosen to avoid such coincidences. It suffices to use different large values for those variables, so these tuples (and prefixes) will be called *far-field tuples* (and *far-field prefixes*).

Thus, when considering equivalence of the GDDs in Fig. 9.a and Fig. 9.b, we would only consider prefixes where $x_{10} \neq x_{20}$, and when considering equivalence of the GDDs in Fig. 9.b and Fig. 9.c, we would only consider prefixes where $x_{20} \neq 7$.

Note that, because of rule 6, the label 'x'$_k$ can occur only within the tree under a '*'-labeled edge from a node A at level k. Thus the '*' label at level k acts as a kind of binder for 'x'$_k$, which may be thought of as a 'bound' variable at level k, but is called a 'free' variable wherever it occurs within A['*'].

Let us define, for a GDD A, the set of constants CO_A and the set of (indices of) *free variables* FV_A occurring in A.

Definition of CO and FV: When $A \in R$ is a leaf, $CO_A = \emptyset$, and $FV_A = \emptyset$.

Otherwise, $CO_A = \bigcup_{p \in labels(A) \cup \{\text{'*'}\}} (CO_{A[p]}) \cup \{c | c \in \mathbb{N} \cap labels(A)\}$,

And $FV_A = (\bigcup_{p \in labels(A)} FV_{A[p]}) \cup (FV_{A[\text{'*'}]} \setminus \{k\}) \cup \{i \in \mathbb{N} | \text{'}x\text{'}_i \in labels(A)\}$, where $k = level(A)$

I also define, for a set S of GDDs (all having the same level), the set of far-field prefixes $far(S)$ for S.

$far(S) = \{\langle x_K \dots x_{k+1} \rangle \in \mathbb{N}^{K-k} | \forall i \in FV_S : \forall j \in FV_S \setminus \{i\} : \forall c \in CO_S : x_i \neq x_j \wedge x_i \neq c\}$, where $CO_S = \bigcup_{A \in S} CO_A$, and $FV_S = \bigcup_{A \in S} FV_A$, and k is the shared level for the GDDs in S.

As a special case, for a singleton $\{A\}$, we have:

$far(\{A\}) = \{\langle x_K \dots x_{k+1} \rangle \in \mathbb{N}^{K-k} | \forall i \in FV_A : \forall j \in FV_A \setminus \{i\} : \forall c \in CO_A : x_i \neq x_j \wedge x_i \neq c\}$, where $k = level(A)$.

Thus, in a far-field prefix for S, the value of each free variable never coincides with the value of any constant occurring in S or any other free variable. Since, for any GDD $A \in S$, C_A only has inequality constraints involving constants in CO_A, and variables in FV_A, they are automatically satisfied by any prefix in $far(S)$. Thus, $far(S) \subseteq sat(A)$, for every unambiguous GDD $A \in S$.

I also write $x \in far(S)$ when $x_{\uparrow k} \in far(S)$, in which case I say tuple x is far-field for S.

I now define *far-field equivalence* (which will turn out to be the same as equivalence) as follows:

$\forall A, B \in GDD_k^K, k \leq K$:

$A \underset{ff}{=} B$ iff: $\forall x \in \mathbb{N}^K \cap far(\{A, B\}) : f_A(x) = f_B(x)$.

For a given single GDD, the set of paths chosen by tuples having a far-field prefix is the same as the set of paths chosen by a tuples having a different far-field prefix. In particular, for a GDD $A \in GDD_k^K$, and any tuple x where $x_{\uparrow k} \in far(\{A\})$, for any prefix $y_{\uparrow k} \in far(\{A\})$, there is a suffix $y_{\downarrow k}$, such that tuple $y = y_{\uparrow k} y_{\downarrow k}$ choses the same path through A as does x.

This may be formalized as follows (with respect to values instead of paths):

$$\forall A \in GDD_k^K : \forall x_{\uparrow k} \in far(\{A\}) : \forall x_{\downarrow k} \in \mathbb{N}^k : \forall y_{\uparrow k} \in far(\{A\}) : \exists y_{\downarrow k} \in \mathbb{N}^k : f_A(x_{\uparrow k} x_{\downarrow k}) = f_A(y_{\uparrow k} y_{\downarrow k}) \quad (8)$$

I prefer to prove a more general version of this, but must first define a special relation $(\leq)$ among GDDs.

We say, for any GDDs A, and A', where $level(A) = level(A')$, that $A' \leq A$ iff $CO_{A'} \subseteq CO_A \wedge FV_{A'} \subseteq FV_A$, so that $far(\{A\}) \subseteq far(\{A'\})$.

Given this, we can identify $far(\{A\}) = far(\{A, A'\})$.

I now give the more general statement I wish to prove:

$$\begin{gathered} \forall A \in GDD_k^K : \\ \forall x_{\uparrow k} \in far(\{A\}) : \\ \forall x_{\downarrow k} \in \mathbb{N}^k : \\ \forall y_{\uparrow k} \in far(\{A\}) : \end{gathered}$$

$$\exists y_{\downarrow k} \in \mathbb{N}^k :$$

$$\forall A' \leq A : f_{A'}(x_{\uparrow k} x_{\downarrow k}) = f_{A'}(y_{\uparrow k} y_{\downarrow k}) \tag{9}$$

To prove this, I must show a method for finding a $y_{\downarrow k}$ that ensures $f_{A'}(x_{\uparrow k} x_{\downarrow k}) = f_{A'}(y_{\uparrow k} y_{\downarrow k})$, given the values of A, $x_{\uparrow k}$, $x_{\downarrow k}$, and $y_{\uparrow k}$, in a way that is mostly independent of A'.

Note that when $j \in FV_{A'}$ and $c \in CO_{A'}$ we have $x_j \neq c \wedge y_j \neq c$, since we are given $x_{\uparrow k} \in far(\{A\})$ and $y_{\uparrow k} \in far(\{A\})$. The far-field nature of $x_{\uparrow k}$ and $y_{\uparrow k}$ also provides $(x_{\uparrow k})_i \neq (x_{\uparrow k})_j$ and $(y_{\uparrow k})_i \neq (y_{\uparrow k})_j$ whenever $i, j \in FV_{A'}$ since we are given $FV_{A'} \subseteq FV_A$.

Consider that $f_{A'}(x) = f_{A'}(x_{\uparrow k} x_{\downarrow k})$ depends only on the destination of the path through A chosen by $x_{\downarrow k}$. That path, in turn, depends only on which of certain equations holds. The relevant equations are all in the following classes:

1. $x_i = x_j$ for some $j \in FV_{A'} \subseteq FV_A$, and $i \in \{1, \ldots, k\}$
2. $x_i = x_j$ for some $j \in \{1, \ldots, k\}$, and $i \in \{1, \ldots, k\}$
3. $x_i = c$ for some $c \in CO_{A'} \subseteq CO_A$, and $i \in \{1, \ldots, k\}$

Note that the following case:

4. $x_i = x_j$ for some $j \in \{k+1, \ldots, K\} \setminus FV_A$, and $i \in \{1, \ldots, k\}$

is irrelevant, since such variables x_j do not occur in any A' such that $FV_{A'} \subseteq FV_A$.

Thus, my method needs only to find a $y_{\downarrow k}$ for which each of the following requirements holds:

1. $(y_{\downarrow k})_i = (y_{\uparrow k})_j \Leftrightarrow (x_{\downarrow k})_i = (x_{\uparrow k})_j$ for all $j \in FV_A$, and $i \in \{1, \ldots, k\}$
2. $(y_{\downarrow k})_i = (y_{\downarrow k})_j \Leftrightarrow (x_{\downarrow k})_i = (x_{\downarrow k})_j$ for all $j \in \{1, \ldots, k\}$, and $i \in \{1, \ldots, k\}$
3. $(y_{\downarrow k})_i = c \Leftrightarrow (x_{\downarrow k})_i = c$ for all $c \in CO_A$, and $i \in \{1, \ldots, k\}$

My method will produce each of the values in $y_{\downarrow k}$ by a transformation of the corresponding values in $x_{\downarrow k}$. So, my method produces $y_{\downarrow k}$ via some function g, so that $(y_{\downarrow k})_i = g((x_{\downarrow k})_i)$ for all $i \in \{1, \ldots, k\}$.

Due to requirement 3, we have $\forall c \in CO_A : g(c) = c$, while requirement 1 provides $\forall j \in FV_A : g((x_{\uparrow k})_j) = (y_{\uparrow k})_j$, and these two cases involve mutually exclusive domains due to the far-field nature of the prefixes. The remaining portion $D = \{x_i : i \in \{1, \ldots, k\}\} \setminus \{x_j : j \in FV_A\} \setminus CO_A$ of the domain of g can be handled in many ways if $\mathbb{N} \setminus CO_A$ is infinite (which will always be the case in subsequent proofs) by mapping each member of D to a unique natural not in CO_A or in either prefix.

As there is always such a g meeting the requirements 1, 2, and 3, g may be used to produce the required suffix $y_{\downarrow k}$ satisfying Equation 9 in all cases. Thus Equation 9 is proved, and so is Equation 8, as a special case.

Now recall the definition of far-field equivalence:
$\forall A, B \in GDD_k^K, k \leq K :$
$A \underset{ff}{=} B$ iff: $\forall x \in \mathbb{N}^K \cap far(\{A, B\}) : f_A(x) = f_B(x)$.
Since far depends only on prefixes, this may be re-written as:
$\forall A, B \in GDD_k^K, k \leq K : A \underset{ff}{=} B \Leftrightarrow \forall x_{\uparrow k} \in far(\{A, B\}) : \forall x_{\downarrow k} \in \mathbb{N}^k : f_A(x_{\uparrow k} x_{\downarrow k}) = f_B(x_{\uparrow k} x_{\downarrow k})$.
This implies:
$\forall A, B \in GDD_k^K, k \leq K : A \underset{ff}{\neq} B \Leftrightarrow \exists x_{\uparrow k} \in far(\{A, B\}) : \exists x_{\downarrow k} \in \mathbb{N}^k : f_A(x_{\uparrow k} x_{\downarrow k}) \neq f_B(x_{\uparrow k} x_{\downarrow k})$
Specializing to the case where $B \leq A$, we have Equation 10:

$$\forall A, B \in GDD_k^K, B \leq A, k \leq K : A \underset{ff}{\neq} B \Leftrightarrow \exists x_{\uparrow k} \in far(\{A, B\}) : \exists x_{\downarrow k} \in \mathbb{N}^k : f_A(x_{\uparrow k} x_{\downarrow k}) \neq f_B(x_{\uparrow k} x_{\downarrow k}) \tag{10}$$

Within this statement, I would like to manipulate the body ($f_A(x_{\uparrow k} x_{\downarrow k}) \neq f_B(x_{\uparrow k} x_{\downarrow k})$) using Equation 9. Equation 9 must first be instantiated as follows: Let the variable A from Equation 9 be A from Equation 10, so that $far(A) = far(\{A, B\})$.

Equation 9 thus becomes:

$$\begin{gathered} \forall x_{\uparrow k} far(\{A, B\}) : \\ \forall x_{\downarrow k} \in \mathbb{N}^k : \\ \forall y_{\uparrow k} \in far(\{A, B\}) : \\ \exists y_{\downarrow k} \in \mathbb{N}^k : \\ \forall A' \leq A : f_{A'}(x_{\uparrow k} x_{\downarrow k}) = f_{A'}(y_{\uparrow k} y_{\downarrow k}) \end{gathered}$$

in this context.

Specializing the $\forall A' \leq A$ for A and for B, yields:

$$\begin{gathered} \forall x_{\uparrow k} \in far(\{A, B\}) : \\ \forall x_{\downarrow k} \in \mathbb{N}^k : \\ \forall y_{\uparrow k} \in far(\{A, B\}) : \\ \exists y_{\downarrow k} \in \mathbb{N}^k : \\ f_A(x_{\uparrow k} x_{\downarrow k}) = f_A(y_{\uparrow k} y_{\downarrow k}) \wedge f_B(x_{\uparrow k} x_{\downarrow k}) = f_B(y_{\uparrow k} y_{\downarrow k}) \end{gathered}$$

Applying this into Equation 10 implies:
$\forall A, B \in GDD_k^K, B \leq A, k \leq K : A \underset{ff}{\neq} B \Rightarrow$
$\exists x_{\uparrow k} \in far(\{A, B\}) : \exists x_{\downarrow k} \in \mathbb{N}^k :$
$\forall y_{\uparrow k} \in far(\{A, B\}) : \exists y_{\downarrow k} \in \mathbb{N}^k :$
$f_A(x_{\uparrow k} x_{\downarrow k}) = f_A(y_{\uparrow k} y_{\downarrow k}) \wedge f_A(x_{\uparrow k} x_{\downarrow k}) \neq f_B(x_{\uparrow k} x_{\downarrow k}) \wedge f_B(x_{\uparrow k} x_{\downarrow k}) = f_B(y_{\uparrow k} y_{\downarrow k})$,
which, then implies Equation 11:

$$\forall A, B \in GDD_k^K, B \leq A, k \leq K : A \underset{ff}{\neq} B \Rightarrow \forall y_{\uparrow k} \in far(\{A, B\}) : \exists y_{\downarrow k} \in \mathbb{N}^k : f_A(y_{\uparrow k} y_{\downarrow k}) \neq f_B(y_{\uparrow k} y_{\downarrow k}) \tag{11}$$

For use in Section A.3, we need the following additional notation: I will write $A - p$ (for an unambiguous GDD A and an edge label p), to mean a GDD A' where $labels(A') = labels(A) \setminus \{p\}$, and for every label q in $labels(A') \cup \{\text{`*'}\}$, $A'[q] = A[q]$. Thus $A - p$ is a GDD node just like $A$ except it is missing the edge labeled $p$ and the subtree $A[p]$ under it. Note that $A - p$ is unambiguious iff $(\text{`}x\text{'}_{level(A)} \neq p) \notin C_{A[\text{`*'}]}$.

A.3 Canonicity of GDDs

We say an encoding method E, decoded by a function f, is *canonical* over a domain D iff for each d in D there is only one encoding e in E such that $d = f_e$.

The following canonicity rules suffice to ensure canonicity of GDDs:

1. Rules 1-6 in Section 3.1.1 are satisfied, ensuring unambiguous definition of f.

2. For any node $A \in GDD_k^K$, and any label $p \in labels(A)$, we have either $(p \neq x_k) \in C_{A[\text{`*'}]}$ or $\neg(A \underset{ff}{=} A - p)$.

   This rule forces normalized nodes to be 'minimal' in the sense that each outgoing edge, labeled $p \neq$ '*', should have its existence justified, either by a constraint $(p \neq x_k)$ in $C_{A[\text{`*'}]}$, which, by rule 5 requires the edge p to exist, or by the fact that A could not be effectively replaced by $A - p$.

Henceforth all GDDs are assumed to be normalized (they follow these canonicity rules). When referring to possibly un-normalized diagrams, the symbol *UGDD* will be used. I proceed now to show that normalization enforces canonicity.

The canonicity of ('top level') GDDs is stated formally as follows:

$$\forall A, B \in GDD_K^K : A \underset{f}{=} B \Rightarrow A = B \tag{12}$$

This is a special case of canonicity of ('any level') GDDs:

$$\forall k \leq K : \forall A, B \in GDD_k^K : A \underset{f}{=} B \Rightarrow A = B \tag{13}$$

where $k = K$.

This may be shown as a consequence
of (1): $\forall k \leq K : \forall A, B \in GDD_k^K : A \underset{f}{=} B \Rightarrow A \underset{ff}{=} B$
and (2): $\forall k \leq K : \forall A, B \in GDD_k^K : A \underset{ff}{=} B \Rightarrow A = B$

(1) is obvious from the definitions above of functional equivalence and far-field equivalence

The contrapositive of (2):
$\forall k \leq K : \forall A, B \in GDD_k^K : A \neq B \Rightarrow (A \underset{ff}{\neq} B)$
is a consequence of:

$$\forall k \leq K : \forall A, B \in GDD_k^K : \forall x_{\uparrow k} \in far(\{A, B\}) : A \neq B \Rightarrow \exists x_{\downarrow k} f_A(x_{\uparrow k} x_{\downarrow k}) \neq f_B(x_{\uparrow k} x_{\downarrow k}) \tag{14}$$

which is proven by induction on level, as follows:

In the base case, where $A, B \in R \wedge k = 0$, the proof is trivial since only Equation 4 applies, and $A = f_A(x)$ and $B = f_B(x)$ regardless of the value of x, so that $A \neq B \Rightarrow \exists x_{\downarrow k}\ f_A(x_{\uparrow k} x_{\downarrow k}) \neq f_B(x_{\uparrow k} x_{\downarrow k})$, is satisfied by letting the suffix $x_{\downarrow k}$ be the empty tuple.

In the inductive case, for some $k < K$, we are given:

$$\forall A, B \in GDD_k^K : \forall x_{\uparrow k} \in far(\{A, B\}) : A \neq B \Rightarrow \exists x_{\downarrow k} : f_A(x_{\uparrow k} x_{\downarrow k}) \neq f_B(x_{\uparrow k} x_{\downarrow k}) \tag{15}$$

and must prove:

$$\forall A, B \in GDD_{k+1}^K : \forall x_{\uparrow k+1} \in far(\{A, B\}) : A \neq B \Rightarrow \exists x_{\downarrow k+1} : f_A(x_{\uparrow k+1} x_{\downarrow k+1}) \neq f_B(x_{\uparrow k+1} x_{\downarrow k+1}) \tag{16}$$

Based on the possible values of A and B in Equation 16, the following cases occur:

1. $A = B$

   In this case, the result trivially holds.

2. $A \neq B$ and $A[c] \neq B[c] \wedge A\overrightarrow{c} \wedge B\overrightarrow{c}$ for some $c \in \mathbb{N}$.

   Choose $x_{k+1} = c$. Let $x_{\uparrow k} = x_{\uparrow k+1}x_{k+1} = x_{\uparrow k+1}c$. Since $c \in CO_A \cap CO_B \wedge x_{\uparrow k+1} \in far(\{A, B\})$ and $(k+1) \notin (FV_{A[c]} \cup FV_{B[c]})$ (due to rule 6), we know $x_{\uparrow k} \in far(\{A[c], B[c]\})$. Obviously $A[c] \in GDD_k^K$ and $B[c] \in GDD_k^K$, so we can apply Equation 15 (the inductive hypothesis) to show that $\exists x_{\downarrow k} : f_{A[c]}(x_{\uparrow k}x_{\downarrow k}) \neq f_{B[c]}(x_{\uparrow k}x_{\downarrow k})$. Let $x_{\downarrow k}$ be the suffix that satisfies Equation 15 in this case, and let $x = x_{\uparrow k+1}cx_{\downarrow k}$, so that we consequently have $f_{A[c]}(x) \neq f_{B[c]}(x)$. In evaluating $f_A(x)$ and $f_B(x)$, Equation 5 applies to both, so we have $f_A(x) = f_{A[c]}(x)$ and $f_B(x) = f_{B[c]}(x)$, so that, when $x_{\downarrow k+1} = cx_{\downarrow k}$, we have $f_A(x) \neq f_B(x)$, the required result.

3. $A \neq B$ and $A[\text{`x'}_i] \neq B[\text{`x'}_i] \wedge A\overrightarrow{\text{`x'}_i} \wedge B\overrightarrow{\text{`x'}_i}$ for some $i > k+1$.

   Choose $x_{k+1} = x_i$. Let $x_{\uparrow k} = x_{\uparrow k+1}x_{k+1} = x_{\uparrow k+1}x_i$. Since $i \in FV_A \cap FV_B \wedge x_{\uparrow k+1} \in far(\{A, B\})$ and $(k+1) \notin (FV_{A[\text{`x'}_i]} \cup FV_{B[\text{`x'}_i]})$ (due to rule 6), we know $x_{\uparrow k} \in far(\{A[\text{`x'}_i], B[\text{`x'}_i]\})$. Obviously $A[\text{`x'}_i] \in GDD_k^K$ and $B[\text{`x'}_i] \in GDD_k^K$, so we can apply Equation 15 to show that $\exists x_{\downarrow k} : f_{A[\text{`x'}_i]}(x_{\uparrow k}x_{\downarrow k}) \neq f_{B[\text{`x'}_i]}(x_{\uparrow k}x_{\downarrow k})$. Let $x_{\downarrow k}$ be the suffix that satisfies Equation 15 in this case, and let $x = x_{\uparrow k+1}x_ix_{\downarrow k}$, so that we consequently have $f_{A[\text{`x'}_i]}(x) \neq f_{B[\text{`x'}_i]}(x)$. In evaluating $f_A(x)$ and $f_B(x)$, Equation 5 applies to both, so we have $f_A(x) = f_{A[\text{`x'}_i]}(x)$ and $f_B(x) = f_{B[\text{`x'}_i]}(x)$, so that we have $f_A(x) \neq f_B(x)$, the required result.

4. $A \neq B$ and $A[\text{`*'}] \neq B[\text{`*'}]$

   Choose $x_{k+1} = n \in \mathbb{N}$ an integer not present in $x_{\uparrow k+1}$ or in CO_A or in CO_B, so $n \notin (CO_A \cup CO_B)$, and $n \neq x_i$ for all $i > k+1$. Let $x_{\uparrow k} = x_{\uparrow k+1}x_{k+1} = x_{\uparrow k+1}n$. Since $x_{\uparrow k+1} \in far(\{A, B\})$, we also have $x_{\uparrow k} \in far(\{A[\text{`*'}], B[\text{`*'}]\})$. Obviously $A[\text{`*'}] \in GDD_k^K$ and $B[\text{`*'}] \in GDD_k^K$, so we can apply Equation 15 to show that $\exists x_{\downarrow k} : f_{A[\text{`*'}]}(x_{\uparrow k}x_{\downarrow k}) \neq f_{B[\text{`*'}]}(x_{\uparrow k}x_{\downarrow k})$. Let $x_{\downarrow k}$ be the suffix that satisfies Equation 15 in this case, and let $x = x_{\uparrow k+1}nx_{\downarrow k}$, so that we consequently have $f_{A[\text{`*'}]}(x) \neq f_{B[\text{`*'}]}(x)$. In evaluating $f_A(x)$ and $f_B(x)$, Equation 7 applies to both, so we have $f_A(x) = f_{A[\text{`*'}]}(x)$ and $f_B(x) = f_{B[\text{`*'}]}(x)$, so that we have $f_A(x) \neq f_B(x)$, the required result.

5. $A \neq B \wedge A[\text{`*'}] = B[\text{`*'}]$ and $x_i = x_j \wedge A\overrightarrow{\text{`x'}_i} \wedge \neg B\overrightarrow{\text{`x'}_i} \wedge B\overrightarrow{\text{`x'}_j} \wedge \neg A\overrightarrow{\text{`x'}_j}$ for some $i, j > k$ where $i \neq j$.

   This case cannot occur due to the definition of far-field paths. Since $\{i, j\} \subseteq FV_A \cup FV_B$, we have $i \neq j \Rightarrow x_i \neq x_j$ for all prefixes $x_{\uparrow k+1}$ in $far(\{A, B\})$.

6. $A \neq B \wedge A[\text{`*'}] = B[\text{`*'}]$ and $c = x_j \wedge A\overrightarrow{c} \wedge \neg B\overrightarrow{c} \wedge B\overrightarrow{\text{`x'}_j} \wedge \neg A\overrightarrow{\text{`x'}_j}$ for some $i > k$ and some constant c.

   This case cannot occur due to the definition of far-field paths. Since $c \in CO_A$ and $j \in FV_B$, we have $c \neq x_j$ for all prefixes $x_{\uparrow k+1}$ in $far(\{A, B\})$.

7. $A \neq B \wedge A[\text{`*'}] = B[\text{`*'}]$ and $x_i = c \wedge A\overrightarrow{\text{`x'}_i} \wedge \neg B\overrightarrow{\text{`x'}_i} \wedge B\overrightarrow{c} \wedge \neg A\overrightarrow{c}$ for some $j > k$ and some constant c.

   This case can be reduced to case 6, by swapping A with B and so cannot occur.

8. $A \neq B \wedge A[\text{`*'}] = B[\text{`*'}]$ and $A\overrightarrow{p} \wedge \neg B\overrightarrow{p}$ for some edge label $p \neq$ `*'

   By canonicity rule 2 applied to A, we have either $(p \neq x_k) \in C_{A[\text{`*'}]}$ or $\neg(A \underset{ff}{=} A - p)$. But $(p \neq x_k) \in C_{A[\text{`*'}]}$ cannot be the case, since we would have $(p \neq x_k) \in C_{B[\text{`*'}]}$ because $A[\text{`*'}] = B[\text{`*'}]$, and this would violate ambiguity rule 5 since we have $\neg B\overrightarrow{p}$.

   Hence $A - p$ is unambiguous and we have $\neg(A \underset{ff}{=} A - p)$.

Obviously $A - p \leq A$, so that $far(\{A, A-p\}) = far(\{A\})$, and then Equation 11 provides:

$\forall y_{\uparrow k+1} \in far(\{A\}) : \exists y_{\downarrow k+1} \in \mathbb{N}^{k+1} : f_A(y_{\uparrow k+1} y_{\downarrow k+1}) \neq f_{A-p}(y_{\uparrow k+1} y_{\downarrow k+1})$, which implies:

$\forall y_{\uparrow k+1} \in far(\{A, B\}) : \exists y_{\downarrow k+1} \in \mathbb{N}^{k+1} : f_A(y_{\uparrow k+1} y_{\downarrow k+1}) \neq f_{A-p}(y_{\uparrow k+1} y_{\downarrow k+1})$.

Changing the names of the quantified variables for application here yields:

$\forall x_{\uparrow k+1} \in far(\{A, B\}) : \exists x_{\downarrow k+1} \in \mathbb{N}^{k+1} : f_A(x_{\uparrow k+1} x_{\downarrow k+1}) \neq f_{A-p}(x_{\uparrow k+1} x_{\downarrow k+1})$.

Here it is clear that, in the value of any $x_{\downarrow k+1} = x_{k+1} x_{\downarrow k}$, which satisfies this equation, that x_{k+1} must be $eval_{x_{\uparrow k+1}}(p)$, as any other value would produce $f_A(x_{\uparrow k+1} x_{\downarrow k+1}) = f_{A-p}(x_{\uparrow k+1} x_{\downarrow k+1})$, since A and $A - p$ have the same edge labels and children, except for p with child $A[p]$.

Hence, we choose $x_{k+1} = eval_{x_{\uparrow k+1}}(p)$, so $x_{\downarrow k+1} = x_{k+1} x_{\downarrow k} = (eval_{x_{\uparrow k+1}}(p)) x_{\downarrow k}$.

Thus, $f_{A-p}(x_{\uparrow k+1} x_{k+1} x_{\downarrow k}) = f_{(A-p)[\text{`*'}]}(x_{\uparrow k+1} x_{k+1} x_{\downarrow k}) = f_{A[\text{`*'}]}(x_{\uparrow k+1} x_{k+1} x_{\downarrow k})$, where $x_{k+1} = eval_{x_{\uparrow k+1}}(p)$, since only Equation 7 applies to evaluation of $f_{A-p}(x_{\uparrow k+1} x_{k+1} x_{\downarrow k})$.

But $A[\text{`*'}] = B[\text{`*'}]$, and B also lacks an edge labeled p, so that only Equation 7 applies to evaluation of $f_B(x_{\uparrow k+1} x_{\downarrow k+1})$ (One might imagine that B could have some other edge labeled q, where q would match x_{k+1} ($eval_{x_{\uparrow k+1}}(q) = x_{k+1}$); however, such situations are covered in the imaginary cases 5, 6, and 7, above). Consequently, $f_B(x_{\uparrow k+1} x_{\downarrow k+1}) = f_{B[\text{`*'}]}(x_{\uparrow k+1} x_{\downarrow k+1}) = f_{A[\text{`*'}]}(x_{\uparrow k+1} x_{k+1} x_{\downarrow k}) = f_{A-p}(x_{\uparrow k+1} x_{\downarrow k+1})$.

Hence we have $f_B(x_{\uparrow k+1} x_{\downarrow k+1}) = f_{A-p}(x_{\uparrow k+1} x_{\downarrow k+1})$ and $f_A(x_{\uparrow k+1} x_{\downarrow k+1}) \neq f_B(x_{\uparrow k+1} x_{\downarrow k+1})$, the required result.

9. $A \neq B \wedge A[\text{`*'}] = B[\text{`*'}]$ and $B\overrightarrow{p} \wedge \neg A\overrightarrow{p}$ for some edge label $p \neq$ `*'

   This reduces to case 8 by switching A with B.

Thus, Equation 14 is inductively proven.
Hence, by the reasoning above, normalized GDDs are canonical.

A.4 Bundles and BundleUnions

A Bundle of length K is a non-empty set $\{X \in \mathbb{N}^K | C_1(X) \wedge \ldots \wedge C_n(X)\}$ of K-tuples, where each of the n ($n \in \mathbb{N}$) constraints C_i ($i \in \{1, \ldots, n\}$) has the form: "$A = B$", or the form: "$A \neq B$", where each of A and B has the form "X_k", for some $k \in \{1, \ldots, K\}$, or the form "c" for some $c \in \mathbb{N}$. It is obvious that a non-empty intersection of Bundles is also a Bundle.

A BundleUnion of length K is a union of zero or more Bundles.

It is obvious that BundleUnions are closed over union, intersection, complement and cartesian products.

A Bundle in *normal form* is written with certain limits on the way its constraints are written as follows:

1. Each constraint has one of these forms: "$X_k = X_j$", "$X_k = c$", "$X_k \neq X_j$", or "$X_k \neq c$", for some $j > k, \{j, k\} \in \{1, \ldots, K\}, c \in \mathbb{N}$.

2. For each $k \in \{1, \ldots, K\}$, one of the following holds:

   (a) There is exactly one constraint with the form: "$X_k = B$", and there are no constraints having any of the forms: "$X_k \neq B$", "$A = X_k$", or "$A \neq X_k$".

   (b) There are no constraints with the form: "$X_k = B$".

It is obvious that any Bundle can be written in normal form. Any written form of Bundle can be converted to normal form by re-writing constraints that violate the above rules as follows:

1. Each constraint has one of these forms: "$X_k = X_j$", "$X_k = c$", "$X_k \neq X_j$", or "$X_k \neq c$", for some $j > k, \{j, k\} \in \{1, \ldots, K\}, c \in \mathbb{N}$. This can be enforced through the following re-writing:
   (a) A constraint of one of the forms: "$c_1 \neq c_2$", "$c = c$", or "$X_k = X_j$", with $k = j$,$c_1 \neq c_2$, may be removed.
   (b) A constraint of one of the forms: "$c_1 = c_2$", "$c \neq c$", or "$X_k \neq X_j$", with $k = j$,$c_1 \neq c_2$, cannot exist, as a Bundle is non-empty.
   (c) A constraint of one of the forms: "$X_k = X_j$" or "$X_k \neq X_j$", with $k > j$, may be rewritten as: "$X_j = X_k$" or "$X_j \neq X_k$", respectively.
   (d) A constraint of one of the forms: "$c = X_k$" or "$c \neq X_k$", may be rewritten as: "$X_k = c$" or "$X_k \neq c$", respectively.
2. For each $k \in \{1, \ldots, K\}$, one of the following holds:
   (a) There is exactly one constraint with the form: "$X_k = B$", and there are no constraints having any of the forms: "$X_k \neq B$", "$X_m = X_k$", or "$X_m \neq X_k$", where $k > m$.
   (b) There are no constraints with the form: "$X_k = B$".

   This can be refactored as:

   For each $k \in \{1, \ldots, K\}$, both of the following hold:
   (a) There is at most one constraint with the form: "$X_k = B$". This can be enforced through the following re-writing:
      i. Constraints: "$X_k = c_1$", "$X_k = c_2$", where $c_1 \neq c_2$, cannot both exist, as a Bundle is non-empty.
      ii. Constraints: "$X_k = c_1$", "$X_k = c_2$", where $c_1 = c_2$, can be collapsed to a single constraint.
      iii. Constraints: "$X_k = X_j$", "$X_k = X_i$", where $i = j > k$, can be collapsed to a single constraint.
      iv. Constraints: "$X_k = X_j$", "$X_k = X_i$", where $i > j > k$, can be re-written as: "$X_k = X_i$", "$X_j = X_i$".
      v. Constraints: "$X_k = c$", "$X_k = X_j$", where $j > k$, can be re-written as: "$X_k = X_j$", "$X_j = c$".
   (b) For each constraint with the form: "$X_k = B$", there are no constraints having any of the forms: "$X_k \neq B$", "$X_m = X_k$", or "$X_m \neq X_k$", where $k > m$. This can be enforced through the following re-writing:
      i. Constraints: "$X_k = c$", "$X_k \neq c$" cannot both exist, as a Bundle is non-empty.
      ii. Constraints: "$X_k = c_1$", "$X_k \neq c_2$", where $c_1 \neq c_2$, can be re-written as: "$X_k = c_1$".
      iii. Constraints: "$X_k = c$", "$X_k \neq X_j$", where $j > k$, can be re-written as: "$X_k = c$", "$X_j \neq c$".
      iv. Constraints: "$X_k = X_j$", "$X_k \neq c$", where $j > k$, can be re-written as: "$X_k = X_j$", "$X_j \neq c$".
      v. Constraints: "$X_k = X_j$", "$X_k \neq X_j$" cannot both exist, as a Bundle is non-empty.
      vi. Constraints: "$X_k = X_i$", "$X_k \neq X_j$", where $i > j > k$, can be re-written as: "$X_k = X_i$", "$X_j \neq X_i$".
      vii. Constraints: "$X_k = X_j$", "$X_k \neq X_i$", where $i > j > k$, can be re-written as: "$X_k = X_j$", "$X_j \neq X_i$".

viii. Constraints: "$X_k = B$", "$X_m = X_k$", where $k > m$, can be re-written as: "$X_k = B$", "$X_m = B$", for any B.

ix. Constraints: "$X_k = B$", "$X_m \neq X_k$", where $k > m$, can be re-written as: "$X_k = B$", "$X_m \neq B$", for any B.

Note that those re-writing rules that enforce the second rule never introduce violations of the first rule, and that any new violations of the second rule are for variables with a higher index. Thus, the re-writing is guaranteed to terminate, as long as the first rule is enforced by its rewritings, followed by enforcement of the second rule for constraints of the form $X_0 \ldots$, followed by constraints of the form $X_1 \ldots$, $\ldots$, constraints of the form $X_k \ldots$, after which the constraints will be in normal form.

Note that normalization cannot introduce variable names not present in the non-normal form. Note also that, due to the direction of propagation of changes in the normalization procedure, if a Bundle in normal form is augmented with additional constraints involving no variables with index less than k, then, after normalization, the augmented Bundle has exactly the same constraints on variables with index less than k that the un-augmented *Bundle* had.

A variable X_i is said to be *strongly specified in* a Bundle b iff the following holds: For every $j > i \leq K$:, either the constraint $X_i = X_j$ or the constraint $X_i \neq X_j$ is implied by the constraints in b.

A variable X_i is said to be *strongly specified in* a Bundle b *for* constants $co \subseteq \mathbb{N}$ iff the following holds: X_i is strongly specified in b and: For every $c \in co$, either the constraint $X_i = c$ or the constraint $X_i \neq c$ is implied by the constraints in b.

It is also obvious that the functions encoded by GDDs are Bundle-wise constant functions defined over $\mathbb{N}^K$, where $\mathbb{N}^K$ is partitioned into a finite number of Bundles. Therefore, if all Bundle-wise constant functions defined over $\mathbb{N}^K$ are encodable as GDDs, then GDD encoded characteristic functions are also closed over union, intersection, complement and cartesian products. I show this in the next sub-section.

A.5 GDDs encode (all) Bundle-wise constant functions

It is somewhat obvious that any Bundle-wise constant function defined over $\mathbb{N}^K$ can be encoded as a GDD, however, I will sketch a proof here.

Such an encoding problem is a special case of the problem of encoding a function F_k defined over B_k^K, where $k < K$ and satisfying the following additional conditions:

1. F_k is 'Bundle-wise constant': There is a partition P_k of B_k^K composed of Bundles (that is, each $p \in P_k$ is a Bundle and $p \subseteq B_k^K$, and $\forall p, p' \neq p \in P_k : p \cap p' = \emptyset$, and $\bigcup P_k = B_k^K$), and F_k is constant over each element $p \in P_k$.

2. In all Bundles of P_k, in normal form, the set of constraints, with variables on the left hand side having index $i > k$, are identical.

3. B_k^K has constraints only on variables X_i with $K \geq i > k$

4. every variable X_i with $i > k$ is strongly specified in B_k^K for con_k, and is also strongly specified in every Bundle of P_k for con_k.

Given a Bundle B_k^K, a finite partition P_k of B_k^K into Bundles, and a Bundle-wise constant function $F_k \in B_k^K \to \mathbb{N}$, represented as a mapping $M \in P_k \to \mathbb{N}$ from bundles, which meet the conditions 1-4 above, the following algorithm constructs the encoding of F_k into an UGDD F_{kUGDD}.

Assume (as an inductive hypothesis when $k > 0$) that we already have an algorithm A_{k-1} which, for any Bundle B^K_{k-1}, partition P_{k-1}, and function $F_{k-1} \in B^K_{k-1} \to \mathbb{N}$ also meeting the conditions 1-4 above, substituting $k-1$ for k wherever it occurs, will return the encoding of of F_{k-1} into an UGDD $F_{k-1UGDD}$.

Let con_k be the set of all constants c where either the constraint $X_i = c$ or the constraint $X_i \neq c$ (for some $i \leq k$) occurs in the normal form of any element of P_k. Let fv_k be the set of all 'higher' variable indices $i > k \in \mathbb{N}$ where b in normal form has no constraint either of the form $X_i = X_j$ (for any j) or of the form $X_i = c$ (for any c). Variables X_i where b has such a constraint are bound to the value of X_j, or to the value of c, respectively.

1. $k = K$ so there are no variables X_i with $i > k$, the root case.

   Proceed as in the node case that immediately follows, first noting that in this case there are no 'higher' variables (with index $i > k$). This establishes the invariant conditions 1-4 trivially, since the conditions relate primarily to 'higher' variables. Obviously, B^K_k has no constraints on variables X_i with $i \leq k$, since $B^K_k = B^K_K = \mathbb{N}^K$ in this case.

2. $K > 0$, The node case.

   Assume the invariants established in the root case, which we will propagate to recursive invocations of this procedure.

   Let the constants in con_k be indexed by $1, \ldots, |con_k|$, so that the members of con_k may be referred to as $c_1, \ldots, c_{|con_k|}$.

   Let the (free) variables in fv_k be indexed by $1, \ldots, |fv_k|$, so that the members of fv_k may be referred to as $v_1, \ldots, v_{|fv_k|}$.

   Consider the following partition $Q_k = \{q_{c_1}, \ldots, q_{c_{|con_k|}}, q_{v_1}, \ldots, q_{v_{|fv_k|}}, q_*\}$ of B^K_k according to the 'free' variables and constants mentioned in the constraints of the partition elements of P_k: The $|con_k| + |fv_k| + 1$ elements of Q_k are defined as follows:

   For each $i \in \{1, \ldots, |con_k|\}$, Let $q_{c_i} = \{X \in B^K_k | X_k = c_i\}$. Since 'higher' variables in B^K_k are strongly specified in B^K_k for con_k, we automatically have: $q_{c_i} = \{X \in B^K_k | X_k \neq c_1 \wedge \ldots \wedge X_k \neq c_{i-1} \wedge X_k = c_i \wedge X_k \neq c_{i+1} \wedge \ldots \wedge X_k \neq c_{|con_k|} \wedge X_k \neq v_1 \wedge \ldots \wedge X_k \neq v_{fv_k}\}$.

   For each $i \in \{1, \ldots, |fv_k|\}$, Let $q_{v_i} = \{X \in B^K_k | X_k = v_i\}$. Since 'higher' variables in B^K_k are strongly specified in B^K_k for con_k, we automatically have: $q_{c_i} = \{X \in B^K_k | X_k \neq c_1 \wedge \ldots \wedge X_k \neq c_{|con_k|} \wedge X_k \neq v_1 \wedge \ldots \wedge X_k \neq v_{i-1} \wedge X_k = v_i \wedge X_k \neq v_{i+1} \wedge \ldots \wedge X_k \neq v_{fv_k}\}$.

   Let $q_* = \{X \in B^K_k | X_k \neq c_1 \wedge \ldots \wedge X_k \neq c_{|con_k|} \wedge X_k \neq v_1 \wedge \ldots \wedge X_k \neq v_{fv_k}\}$.

   It is obvious from these definitions and their automatic consequences that Q_k is a partition of B^K_k. Since the bound variables are all bound to either a constant or to a free variable, it is also obvious that in each partition element $q_{[]}$ X_k is strongly specified in $q_{[]}$ for con_k (and for any subset of con_k, as well). Note that in q_*, the variable X_k is free, while in every other element of the partition Q_k, X_k is bound.

   In this case, the function is encoded as a non-leaf node having edges labeled with the constants $c_1, \ldots, c_{|con_k|}$, free variables $v_1, \ldots, v_{|fv_k|}$, and $*$, used as indices of the elements $q_{[]}$ of Q_k, where, for each index $t \in \{c_1, \ldots, c_{|con_k|}, v_1, \ldots, v_{|fv_k|}, *\}$, the edge labeled t leads to an UGDD, constructed by A_{k-1}, that encodes the function F_k, with its domain restricted to q_t (which will be known as B^K_{k-1} in the recursive invocation). What remains to be shown is that the invariants can be propagated to the construction.

   For analysis of the specific UGDD at the edge labeled t, let the following definitions hold:

Let $B^K_{k-1} = q_t$ the domain for the function encoded by the child UGDD. Note that B^K_{k-1} has constraints only on variables X_i with $K \geq i > k-1$, since the construction of q_t involves adding only (a) constraint(s) relating X_k to either a constant or to a 'higher' variable, and normalization only propagates constraints to 'higher' variables. Thus, invariant condition 3 is propagated to the recursive invocation. Obviously, X_k is strongly specified in B^K_{k-1} for con_k, so that invariant condition 4 is also propagated to the recursive invocation, since con_{k-1} will be a subset of con_k.

Let $P_{k-1} = \{p \cap B^K_{k-1} | p \in P_k\} \setminus \{\emptyset\}$ the partition P_k restricted to B^K_{k-1}. Each member is a Bundle, as Bundles are closed over intersection.

Let F_{k-1} be F_k with its domain restricted to B^K_{k-1}. F_{k-1} is obviously Bundle-wise constant for the elements of P_{k-1}, so invariant condition 1 is propagated to the recursive invocation.

Each member $p \cap B^K_{k-1}$ of P_{k-1}, starts with a member p of P_k, which by invariant condition 2, all have identical constraints for 'higher' variables. As B^K_{k-1} has constraints only on 'higher' variables and X_k, and normalization propagates constraints toward 'higher' variables, the only possibility for different members of P_{k-1} to have different constraints on 'higher' variables or X_k is if that is caused by different members of P_k having different constraints on X_k. This turns out to not be possible, however, as such constraints will either be redundant with constraints in B^K_{k-1} or contradictory to constraints in B^K_{k-1}, as X_k is strongly specified in B^K_{k-1} for con_k. To see that this is the case, consider some constraint γ, with X_k in the left hand side, within the normal form of a member p of P_k. γ has one of the following forms: (1) $X_k = X_i$ (where $i > k$), (2) $X_k \neq X_i$ (where $i > k$), (3) $X_k = c$ (where $c \in con_k$), (4) $X_k \neq c$ (where $c \in con_k$). Because X_k is strongly specified in B^K_{k-1} for con_k, in each of the four cases, γ either (A) directly contradicts the contradicts B^K_{k-1}, in which case $p \cap B^K_{k-1}$ will be $\emptyset$, and absent from P_{k-1}, hence irrelevant, or, (B) γ is redundant with the constraints in B^K_{k-1}, so that the presence or absence of γ within the normal form of p has no effect on the value of $p \cap B^K_{k-1}$. Thus the constraints on X_k or 'higher' variables within the normal form of p have no effect on the constraints on X_k or 'higher' variables within the normal form of $p \cap B^K_{k-1}$, but may only effect its presence or absence within P_{k-1}. Hence invariant condition 2 is also propagated to the recursive invocation.

With all constraints satisfied for the recursive invocation of A_{k-1}, its correct termination is guaranteed, supplying the desired target of the edge labeled t.

3. $k = 0$, The leaf case.

   There should be exactly one Bundle in P_k, as all variables are 'higher' (with index $i > K$), since the invariants guarantee that all Bundles in P_k are identical. The existence of multiple bundles would contradict the assumption that the Bundles in P_k comprise a partition of B^K_k. The function is then encoded as a leaf node labeled with the range value associated with the unique Bundle in $P_k = P_0$.

The UGDD F_{kUGDD} is obviously unambiguous, satisfying ambiguity rules 1-6 in Section 3.1.1, but may not satisfy canonicity rule 2 in Section A.3. F_{kUGDD} may then be normalized by the a depth-first traversal that removes edges which do not satisfy canonicity rule 2, resulting in a GDD F_{kGDD} encoding the function F_k.

Hence, GDDs encode exactly all Bundle-wise constant functions. Consequently, tuple-sets represented by GDD-encoded characteristic functions are closed over union, intersection, complement, and cartesian products.

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