Non Context-free languages
\[ a^n b^n c^n \]

Context-free languages
\[ w w^R \]

Deterministic Context-free languages
\[ a^n b^n \]

Regular languages
\[ a^* b^* \]
Non-context free languages

\{a^n b^n c^n : n \geq 0\} \quad \{ww : w \in \{a, b\}\}

Context-free languages

\{a^n b^n : n \geq 0\} \quad \{ww^R : w \in \{a, b\}^*\}
The Pumping Lemma for Context-Free Languages
Take an **infinite** context-free language

Generates an infinite number of different strings

Example:

\[
S \rightarrow AB \\
A \rightarrow aBb \\
B \rightarrow Sb \\
B \rightarrow b
\]
\[ S \rightarrow AB \]
\[ A \rightarrow aBb \]
\[ B \rightarrow Sb \]
\[ B \rightarrow b \]

**A derivation:**

\[ S \Rightarrow AB \Rightarrow aBbB \Rightarrow abbB \Rightarrow \]
\[ \Rightarrow abbSb \Rightarrow abbABb \Rightarrow abbaBBb \Rightarrow \]
\[ \Rightarrow abbaBBb \Rightarrow abbaBBb \]
Derivation tree

string $abbabbbb$

$S \Rightarrow AB \Rightarrow aBbB \Rightarrow abbB \Rightarrow$
$\Rightarrow abbSb \Rightarrow abbABb \Rightarrow abbaBbBb \Rightarrow$
$\Rightarrow abbabbBb \Rightarrow abbabbbb$
Derivation tree

\[ \text{string } abbabbbb \]

\[ S \Rightarrow AB \Rightarrow aBbB \Rightarrow abbB \Rightarrow \]
\[ \Rightarrow abbSb \Rightarrow abbABb \Rightarrow abbaBbBb \Rightarrow \]
\[ \Rightarrow abbabbBb \Rightarrow abbabbbb \]
\[ B \Rightarrow Sb \Rightarrow ABb \Rightarrow aBbBb \Rightarrow aBbbb \]

\[ B \Rightarrow b \]
Repeated Part

\[ B \Rightarrow \ldots \Rightarrow aBbb \]
Another possible derivation

\[ B \Rightarrow \ldots \Rightarrow aBbbb \]

\[ B \Rightarrow \ldots \Rightarrow aBbbb \ldots \Rightarrow aaBbbbBbbb \]
\[ S \Rightarrow \ldots \Rightarrow abbaBbbb \]
$S \Rightarrow \ldots \Rightarrow abbaBb$$bb \Rightarrow \ldots \Rightarrow abbaaBb$$bbbbb$
\[ S \Rightarrow \ldots \Rightarrow abbaaBbbbbb \Rightarrow abbaaBbbbbb \]

\[ B \Rightarrow b \]
$S \Rightarrow \ldots \Rightarrow abbaabbbbb$

Therefore, the string

\[ abbaabbbbb \]

is also generated by the grammar
We know:

\[ B \Rightarrow b \]

\[ B \Rightarrow \ldots \Rightarrow aBbb \]

\[ S \Rightarrow \ldots \Rightarrow abbaBbb \]

We also know this string is generated:

\[ S \Rightarrow \ldots \Rightarrow abbaBbb \Rightarrow \quad \]

\[ \Rightarrow abbabbb \]
We know: \[ B \Rightarrow b \]
\[ B \Rightarrow \ldots \Rightarrow aBbbb \]
\[ S \Rightarrow \ldots \Rightarrow abbaBbbb \]

Therefore, this string is also generated:

\[ S \Rightarrow \ldots \Rightarrow abbaBbbb \Rightarrow \]
\[ \Rightarrow abbaaBbbbbb \Rightarrow \]
\[ \Rightarrow abbaabbbbbbb \]
We know: 
\[ B \Rightarrow b \]
\[ B \Rightarrow \ldots \Rightarrow aBbbb \]
\[ S \Rightarrow \ldots \Rightarrow abbaBbbb \]

Therefore, this string is also generated:
\[ S \Rightarrow \ldots \Rightarrow abbaBbbb \Rightarrow \]
\[ \Rightarrow abba(a)^2 B(bbb)^2 bbb \]
\[ \Rightarrow abba(a)^2 b(bbb)^2 bbb \]
We know: \[ B \Rightarrow b \]
\[ B \Rightarrow \ldots \Rightarrow aBbbb \]
\[ S \Rightarrow \ldots \Rightarrow abbaBbbb \]

Therefore, this string is also generated:
\[ S \Rightarrow \ldots \Rightarrow abbaBbbb \Rightarrow \]
\[ \Rightarrow \ldots \]
\[ \Rightarrow abba(a)^i B(bbb)^i bbb \]
\[ \Rightarrow abba(a)^i b(bbb)^i bbb \]
Therefore, knowing that

\[ abbabbbb \]

is generated by grammar \( G \), we also know that

\[ abba(a)^i b(bb)^i bbb \]

is generated by \( G \)
In general:

We are given an infinite context-free grammar \( G \)

Assume \( G \) has no unit-productions
no \( \lambda \)-productions
Take a string \( w \in L(G) \)
with length bigger than

\[ m > \text{(Number of productions)} \times \text{(Largest right side of a production)} \]

Consequence:

Some variable must be repeated in the derivation of \( w \)
$u, v, x, y, z$ : strings of terminals

String $w = uvxyz$

Last repeated variable

repeated
Possible derivations:

* \( S \Rightarrow uAz \)

* \( A \Rightarrow vAy \)

* \( A \Rightarrow x \)
We know:

\[ S \Rightarrow uA z \quad A \Rightarrow vA y \quad A \Rightarrow x \]

This string is also generated:

\[ S \Rightarrow uA z \Rightarrow u x z \]

\[ uv^0 x y^0 z \]
We know:

\[ S \Rightarrow uAz \quad A \Rightarrow vAy \quad A \Rightarrow x \]

This string is also generated:

\[ S \Rightarrow uAz \Rightarrow uvAy \Rightarrow uvxyz \]

The original \[ w = uv^1xy^1z \]
We know:

* \[ S \Rightarrow uAz \]
* \[ A \Rightarrow vAy \]
* \[ A \Rightarrow x \]

This string is also generated:

* * * *
\[ S \Rightarrow uAz \Rightarrow uvAy \Rightarrow uvvAy \Rightarrow uvvxyyz \Rightarrow uvvxyyz \]

\[ uv^2 xy^2 z \]
We know:

\[
\begin{align*}
S & \Rightarrow uAz \\
A & \Rightarrow vAy \\
A & \Rightarrow x
\end{align*}
\]

This string is also generated:

\[
\begin{align*}
S & \Rightarrow uAz \\
& \Rightarrow uvAy_1z_1 \\
& \Rightarrow uvvyAy_2z_2 \\
& \Rightarrow uvvvvAyyzz_3 \\
& \Rightarrow uvvvvyxyyz
\end{align*}
\]

\[uv^3xy^3z\]
We know:

$S \Rightarrow uAz \quad A \Rightarrow vAy \quad A \Rightarrow x$

This string is also generated:

$S \Rightarrow uAz \Rightarrow uvAyz \Rightarrow uvvAyyz \Rightarrow uvvvAyyyy \Rightarrow \ldots$
$\Rightarrow uvvv \ldots vAy \ldots yyyz \Rightarrow$
$\Rightarrow uvvv \ldots vxy \ldots yyyz$

$uv^i x y^i z$
Therefore, any string of the form

$$uv^i xy^i z \quad i \geq 0$$

is generated by the grammar $G$.
Therefore, knowing that $uvxyz \in L(G)$

we also know that $uv^i xy^i z \in L(G)$
Observation: \(|vxy| \leq m\)

Since \(A\) is the last repeated variable
Observation: \[ |vy| \geq 1 \]

Since there are no unit or \( \lambda \) productions
The Pumping Lemma:

For infinite context-free language $L$

there exists an integer $m$ such that

for any string $w \in L$, $|w| \geq m$

we can write $w = uvxyz$

with lengths $|vxy| \leq m$ and $|vy| \geq 1$

and it must be:

$uv^i xy^i z \in L$, for all $i \geq 0$
Applications of The Pumping Lemma
Non-context free languages

\[ \{a^n b^n c^n : n \geq 0\} \]

Context-free languages

\[ \{a^n b^n : n \geq 0\} \]
Theorem: The language

\[ L = \{ a^n b^n c^n : n \geq 0 \} \]

is **not** context free

Proof: Use the Pumping Lemma for context-free languages
Assume for contradiction that $L$ is context-free

Since $L$ is context-free and infinite we can apply the pumping lemma
\[ L = \{a^n b^n c^n : n \geq 0\} \]

Pumping Lemma gives a magic number \( m \)
such that:

Pick any string \( w \in L \) with length \( |w| \geq m \)

We pick: \( w = a^m b^m c^m \)
\[ L = \{a^n b^n c^n : n \geq 0\} \]

\[ w = a^m b^m c^m \]

We can write: \[ w = uvxyz \]

with lengths \[ |vxy| \leq m \quad \text{and} \quad |vy| \geq 1 \]
\[ L = \{a^n b^n c^n : n \geq 0\} \]

\[ w = a^m b^m c^m \]

\[ w = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1 \]

Pumping Lemma says:

\[ uv^i xy^i z \in L \quad \text{for all} \quad i \geq 0 \]
\[ L = \{a^n b^n c^n : n \geq 0\} \]

\[ w = a^m b^m c^m \]

\[ w = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1 \]

We examine all the possible locations of string \(vxy\) in \(w\)
\[ L = \{a^n b^n c^n : n \geq 0\} \]

\[ w = a^m b^m c^m \]

\[ w = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1 \]

**Case 1:** \( vxy \) is within \( a^m \)

\[ \text{aaa...aaa bbb...bbb ccc...ccc} \]

\[ u \quad vxy \quad z \]
L = \{a^n b^n c^n : n \geq 0\}

w = a^m b^m c^m

w = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1

Case 2: \(vxy\) is within \(b^m\)
\[ L = \{ a^n b^n c^n : n \geq 0 \} \]

\[ w = a^m b^m c^m \]

\[ w = u v x y z \quad \mid vxy \mid \leq m \quad \mid vy \mid \geq 1 \]

**Case 3:** \( vxy \) is within \( c^m \)

\[ \begin{array}{c}
m \\
\text{aaa...aaa} \\
u
m \\
\text{bbb...bbb} \\
m \\
\text{ccc...ccc} \\
vxy \\
z
\end{array} \]
\[ L = \{ a^n b^n c^n : n \geq 0 \} \]

\[ w = a^m b^m c^m \]

\[ w = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1 \]

**Case 4:** \( vxy \) overlaps \( a^m \) and \( b^m \)

\[
\begin{array}{c}
\text{aaa...aaa} \\
u
\end{array}
\quad
\begin{array}{c}
\text{bbb...bbb} \\
vxy
\end{array}
\quad
\begin{array}{c}
\text{ccc...ccc} \\
z
\end{array}
\]
\[ L = \{ a^n b^n c^n : n \geq 0 \} \]

\[ w = a^m b^m c^m \]

\[ w = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1 \]

**Case 5:** \( vxy \) overlaps \( b^m \) and \( c^m \)

\[ m \]
\[ \text{aaa...aaa} \quad \text{bbb...bbb} \quad \text{ccc...ccc} \]
\[ u \quad vxy \quad z \]
\[ L = \{a^n b^n c^n : n \geq 0\} \]

\[ w = a^m b^m c^m \]

\[ w = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1 \]

**Case 1:** \( vxy \) is within \( a^m \)

\[ \begin{array}{c}
\text{aaa...aaa} \\
u \end{array} \quad \begin{array}{c}
\text{bbb...bbb} \\
vxy \end{array} \quad \begin{array}{c}
\text{ccc...ccc} \\
z \end{array} \]
\[ L = \{ a^n b^n c^n : n \geq 0 \} \]

\[ w = a^m b^m c^m \]

\[ w = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1 \]

**Case 1:** \( v \) and \( y \) consist from only \( a \)

\[
\begin{array}{ccc}
  & m & \\
  u & vxy & z \\
  & m & \\
  aaa...aaa & bbb...bbb & ccc...ccc
\end{array}
\]
\[ L = \{ a^n b^n c^n : n \geq 0 \} \]

\[ w = a^m b^m c^m \]

\[ w = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1 \]

**Case 1:** Repeating \( v \) and \( y \)

\[ k \geq 1 \]

\[ u \quad v^2xy^2 \quad bbb...bbb \quad ccc...ccc \quad z \]
\[ L = \{a^n b^n c^n : n \geq 0\} \]

\[ w = a^m b^m c^m \]

\[ w = u v x y z \quad |vxy| \leq m \quad |vy| \geq 1 \]

**Case 1:** From Pumping Lemma: \( uv^2 xy^2 z \in L \)

\[ k \geq 1 \]

\[ u \quad v^2 xy^2 \quad m \quad m \quad \]

\[ aaaaaa...aaaaaa \quad bbb...bbb \quad ccc...ccc \]

\[ z \]
\[ L = \{ a^n b^n c^n : n \geq 0 \} \]

\[ w = a^m b^m c^m \]

\[ w = uvxyz \quad \left| vxy \right| \leq m \quad \left| vy \right| \geq 1 \]

**Case 1:** From Pumping Lemma: \( uv^2 xy^2 z \in L \)

\[ k \geq 1 \]

However: \( uv^2 xy^2 z = a^{m+k} b^m c^m \notin L \)

**Contradiction!!**
\[ L = \{a^n b^n c^n : n \geq 0\} \]

\[ w = a^m b^m c^m \]

\[ w = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1 \]

**Case 2:** \( vxy \) is within \( b^m \)

\[
\begin{array}{c}
m \\
\text{aaa...aaa} \\
u
\end{array}
\begin{array}{c}
m \\
\text{bbb...bbb} \\
vxy
\end{array}
\begin{array}{c}
m \\
\text{ccc...ccc} \\
z
\end{array}
\]
\[ L = \{a^n b^n c^n : n \geq 0\} \]

\[ w = a^m b^m c^m \]

\[ w = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1 \]

**Case 2:** Similar analysis with case 1

\[
\begin{align*}
\text{aaa...aaa} & \quad \text{bbb...bbb} & \quad \text{ccc...ccc} \\
\text{u} & \quad \text{vxy} & \quad \text{z}
\end{align*}
\]
\[ L = \{a^n b^n c^n : n \geq 0\} \]

\[ w = a^m b^m c^m \]

\[ w = uvxyz \quad |\text{vxy}| \leq m \quad |\text{vy}| \geq 1 \]

**Case 3:** \text{vxy is within} \( c^m \)

\[
\begin{array}{c}
m \\
\text{aaa...aaa} \\
\text{bbb...bbb} \\
\text{ccc...ccc} \\
\text{u} \\
\text{vxy} \\
\text{z}
\end{array}
\]
\[ L = \{a^n b^n c^n : n \geq 0\} \]

\[ w = a^m b^m c^m \]

\[ w = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1 \]

**Case 3:** Similar analysis with case 1

\[
\begin{array}{c}
\text{aaa...aaa} \\
\text{bbb...bbb} \\
\text{ccc...ccc} \\
\text{u} \\
\text{vxy} \\
\text{z}
\end{array}
\]
\[ L = \{a^n b^n c^n : n \geq 0\} \]

\[ w = a^m b^m c^m \]

\[ w = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1 \]

**Case 4:** vxy overlaps \(a^m\) and \(b^m\)
\[ L = \{a^n b^n c^n : n \geq 0\} \]

\[ w = a^m b^m c^m \]

\[ w = uvxyz \quad |vx| \leq m \quad |vy| \geq 1 \]

**Case 4:** Possibility 1: \(v\) contains only \(a\)

\(y\) contains only \(b\)

\[
\begin{array}{c}
m \\
\vdots \\
\text{aaa...aaa} \\
\text{bbb...bbb} \\
\text{ccc...ccc} \\
\end{array}
\]

\[
\begin{array}{c}
u \\
vxy \\
z
\end{array}
\]
\[ L = \{ a^n b^n c^n : n \geq 0 \} \]

\[ w = a^m b^m c^m \]

\[ w = uvxyz \quad |vxym| \leq m \quad |vy| \geq 1 \]

**Case 4:** Possibility 1: \( v \) contains only \( a \)
\( k_1 + k_2 \geq 1 \)
\( m + k_1 \)
\( \underbrace{aaa...aaa} \)
\( u \)

\( v^2 xy^2 \)

\( m + k_2 \)
\( \underbrace{bbb...bbb} \)

\( ccc...ccc \)

\( m \)

\( z \)
\[ L = \{ a^n b^n c^n : n \geq 0 \} \]

\[ w = a^m b^m c^m \]

\[ w = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1 \]

**Case 4:** From Pumping Lemma: \[ uv^2xy^2z \in L \]

\[ k_1 + k_2 \geq 1 \]

\[ u \quad v^2xy^2 \quad m+k_2 \quad m \]

\[ aaa...aaaaaaa bbbbbb...bbb ccc...ccc \]

\[ z \]
\[ L = \{a^n b^n c^n : n \geq 0\} \]

\[ w = a^m b^m c^m \]

\[ w = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1 \]

**Case 4:** From Pumping Lemma: \( uv^2xy^2z \in L \)

\[ k_1 + k_2 \geq 1 \]

**However:** \( uv^2xy^2z = a^{m+k_1}b^{m+k_2}c^m \notin L \)

**Contradiction!!!**
$L = \{ a^n b^n c^n : n \geq 0 \}$

$w = a^m b^m c^m$

$w = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1$

Case 4: Possibility 2: $v$ contains $a$ and $b$

$y$ contains only $b$

$\begin{array}{c}
\text{aaa...aaa} \\
u
\end{array} \quad \begin{array}{c}
\text{bbb...bbb} \\
vxy
\end{array} \quad \begin{array}{c}
\text{ccc...ccc} \\
z
\end{array}$
\[ L = \{ a^n b^n c^n : n \geq 0 \} \]

\[ w = a^m b^m c^m \]

\[ w = uvxyz \quad \mid vxy \mid \leq m \quad \mid vy \mid \geq 1 \]

**Case 4:** Possibility 2: \( v \) contains \( a \) and \( b \)

\( k_1 + k_2 + k \geq 1 \) \quad \text{\( y \) contains only \( b \)}

\[
\begin{align*}
\text{aaa...aaaa} & \quad \text{abbaabb} \quad \text{bbbbbb...b} \quad \text{ccc...ccc} \\
\text{u} & \quad \text{v}^2 \text{xy}^2 & \quad \text{m+k} & \quad \text{m} \\
\end{align*}
\]
\[ L = \{ a^n b^n c^n : n \geq 0 \} \]
\[ w = a^m b^m c^m \]
\[ w = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1 \]

**Case 4: From Pumping Lemma:** \( uv^2xy^2z \in L \)

\[ k_1 + k_2 + k \geq 1 \]

\[ u \quad \overset{m}{aaa...aaaab} \quad \overset{k_1}{aabb} \quad \overset{k_2}{bbbbbb} \quad \overset{m+k}{bbb} \quad \overset{m}{ccc...ccc} \quad z \]
$$L = \{ a^n b^n c^n : n \geq 0 \}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1$$

**Case 4: From Pumping Lemma:** $uv^2xy^2z \in L$

However:

$$k_1 + k_2 + k \geq 1$$

$$uv^2xy^2z = a^m b^{k_1} a^{k_2} b^{m+k} c^m \notin L$$

**Contradiction!!!**
\[ L = \{a^n b^n c^n : n \geq 0\} \]

\[ w = a^m b^m c^m \]

\[ w = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1 \]

**Case 4:** Possibility 3: \(v\) contains only \(a\), \(y\) contains \(a\) and \(b\)

\[
\begin{array}{cccc}
m & \quad & m & \quad & m \\
\underline{aaa...aaa} & \quad & \underline{bbb...bbb} & \quad & \underline{ccc...ccc} \\
u & \quad & vxy & \quad & z
\end{array}
\]
\[ L = \{ a^n b^n c^n : n \geq 0 \} \]

\[ w = a^m b^m c^m \]

\[ w = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1 \]

**Case 4:** Possibility 3: \( v \) contains only \( a \)
\( y \) contains \( a \) and \( b \)

Similar analysis with Possibility 2
\[ L = \{ a^n b^n c^n : n \geq 0 \} \]

\[ w = a^m b^m c^m \]

\[ w = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1 \]

**Case 5:** \( vxy \) overlaps \( b^m \) and \( c^m \)

\[
\begin{array}{c}
\text{aaa...aaa} \\
\text{bbb...bbb} \\
\text{ccc...ccc}
\end{array}
\]

\[
\begin{array}{c}
u \\
vxy \\
z
\end{array}
\]
\[ L = \{ a^n b^n c^n : n \geq 0 \} \]

\[ w = a^m b^m c^m \]

\[ w = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1 \]

**Case 5:** Similar analysis with case 4
There are no other cases to consider

(since \(|vxy| \leq m\), string \(vxy\) cannot

overlap \(a^m\), \(b^m\) and \(c^m\) at the same time)
In all cases we obtained a contradiction

Therefore: The original assumption that

\[ L = \{a^n b^n c^n : n \geq 0\} \]

is context-free must be wrong

**Conclusion:** \( L \) is not context-free
More Applications of The Pumping Lemma
The Pumping Lemma:
For infinite context-free language $L$
there exists an integer $m$ such that
for any string $w \in L$, $|w| \geq m$
we can write $w = uvxyz$
with lengths $|vxy| \leq m$ and $|vy| \geq 1$
and it must be:
$uv^i xy^i z \in L$, for all $i \geq 0$
Theorem: The language 
\[ L = \{ww : w \in \{a,b\}^*\} \]
is not context free

Proof: Use the Pumping Lemma for context-free languages
\[ L = \{ ww : w \in \{a,b\}^* \} \]

Assume for contradiction that \( L \) is context-free

Since \( L \) is context-free and infinite we can apply the pumping lemma
\[ L = \{ww : w \in \{a,b\}^*\} \]

Pumping Lemma gives a magic number \( m \) such that:

Pick any string of \( L \) with length at least \( m \)

we pick: \( a^m b^m a^m b^m \in L \)
\[ L = \{ww : w \in \{a, b\}^*\} \]

We can write: \( a^m b^m a^m b^m = uvxyz \)

with lengths \( |vxy| \leq m \) and \( |vy| \geq 1 \)

Pumping Lemma says:

\[ uv^ixy^iz \in L \quad \text{for all} \quad i \geq 0 \]
\[ L = \{ww : w \in \{a, b\}^* \} \]

\[ a^m b^m a^m b^m = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1 \]

We examine all the possible locations of string \( vxy \) in \( a^m b^m a^m b^m \)
\[ L = \{ww : w \in \{a, b\}^*\} \]

\[ a^m b^m a^m b^m = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1 \]

**Case 1:** \(vxy\) is within the first \(a^m\)
\[ L = \{ww : w \in \{a, b\}^*\} \]

\[ a^m b^m a^m b^m = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1 \]

**Case 2:** \( v \) is in the first \( a^m \)
\( y \) is in the first \( b^m \)
$L = \{ww : w \in \{a, b\}^*\}$

$a^m b^m a^m b^m = uvxyz$ \quad | \quad vy \leq m \quad | \quad vy \geq 1$

**Case 3:** $v$ overlaps the first $a^m b^m$

$y$ is in the first $b^m$
\[ L = \{ww : w \in \{a, b\}^*\} \]
\[ a^m b^m a^m b^m = uvxyz \quad |vx| \leq m \quad |vy| \geq 1 \]

**Case 4:** \( v \) in the first \( a^m \)

y Overlaps the first \( a^m b^m \)
\[ L = \{ww : w \in \{a,b\}^*\} \]

\[ a^m b^m a^m b^m = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1 \]

**Case 1:** vxy is within the first \(a^m\)

\[ v = a^{k_1} \quad y = a^{k_2} \quad k_1 + k_2 \geq 1 \]
\( L = \{ww : w \in \{a, b\}^*\} \)

\[ a^m b^m a^m b^m = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1 \]

**Case 1:** \(vxy\) is within the first \(a^m\)

\[ v = a^{k_1} \quad y = a^{k_2} \quad k_1 + k_2 \geq 1 \]

\[ u \quad v^2 \quad x \quad y^2 \quad z \]

\[ a \ldots a^m \quad b \ldots b \quad a \ldots a^m \quad b \ldots b \]
\[ L = \{ww : w \in \{a,b\}^*\} \]

\[ a^m b^m a^m b^m = uvxyz \quad \mid vxy \mid \leq m \quad \mid vy \mid \geq 1 \]

**Case 1:** \( vxy \) is within the first \( a^m \)

\[ a^{m+k_1+k_2} b^m a^m b^m = uv^2xy^2z \not\in L \]

\[ k_1 + k_2 \geq 1 \]
\[ L = \{ww : w \in \{a,b\}^*\} \]

\[ a^m b^m a^m b^m = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1 \]

**Case 1:** \(vxy\) is within the first \(a^m\)

\[ a^{m+k_1+k_2} b^m a^m b^m = uv^2xy^2z \notin L \]

However, from Pumping Lemma: \(uv^2xy^2z \in L\)

**Contradiction!!!
\[ L = \{ww : w \in \{a,b\}^*\} \]

\[ a^m b^m a^m b^m = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1 \]

**Case 2:** \( v \) is in the first \( a^m \)

\[ y \] is in the first \( b^m \)

\[ v = a^{k_1} \quad y = b^{k_2} \quad k_1 + k_2 \geq 1 \]
\[ L = \{ww : w \in \{a,b\}^*\} \]

\[ a^m b^m a^m b^m = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1 \]

**Case 2:** \( v \) is in the first \( a^m \)

\( y \) is in the first \( b^m \)

\[ v = a^{k_1} \quad y = b^{k_2} \quad k_1 + k_2 \geq 1 \]

\[ u \underline{a} \cdots \underline{a} \underline{b} \cdots \underline{b} \underline{a} \cdots \underline{a} \underline{b} \cdots \underline{b} \underline{z} \]
\[ L = \{ww : w \in \{a,b\}^*\} \]
\[ a^m b^m a^m b^m = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1 \]

**Case 2:** \( v \) is in the first \( a^m \)
\( y \) is in the first \( b^m \)

\[ a^{m+k_1} b^{m+k_2} a^m b^m = uv^2 xy^2 z \not\in L \]

\[ k_1 + k_2 \geq 1 \]
\[ L = \{ ww : w \in \{a, b\}^* \} \]

\[ a^m b^m a^m b^m = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1 \]

**Case 2:** \( v \) is in the first \( a^m \)
\( y \) is in the first \( b^m \)

\[ a^{m+k_1} b^{m+k_2} a^m b^m = uv^2 xy^2 z \notin L \]

However, from Pumping Lemma: \( uv^2 xy^2 z \in L \)

**Contradiction!!!**
\[ L = \{ w \in \{a, b\}^* \} \]

\[ a^m b^m a^m b^m = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1 \]

**Case 3:** \( v \) overlaps the first \( a^m b^m \)

\( y \) is in the first \( b^m \)

\[ v = a^{k_1} b^{k_2} \quad y = b^{k_3} \quad k_1, k_2 \geq 1 \]
\[ L = \{ww : w \in \{a, b\}^*\} \]

\[ a^m b^m a^m b^m = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1 \]

**Case 3:** \(v\) overlaps the first \(a^m b^m\)

\(y\) is in the first \(b^m\)

\[ v = a^{k_1} b^{k_2} \quad y = b^{k_3} \quad k_1, k_2 \geq 1 \]

\[
\begin{array}{cccccc}
  a & \ldots & a & b & \ldots & b \\
  u & & v^2 & x & y^2 & z \\
  a & \ldots & a & b & \ldots & b \\
\end{array}
\]
\[ L = \{ww : w \in \{a, b\}^*\} \]

\[ a^m b^m a^m b^m = uvxyz \quad |vxy| \leq m \quad \left|vy\right| \geq 1 \]

**Case 3:** \(v\) overlaps the first \(a^m b^m\)
\(y\) is in the first \(b^m\)

\[ a^m b^{k_2} a^{k_1} b^{m+k_3} a^m b^m = uv^2 xy^2 z \notin L \]

\(k_1, k_2 \geq 1\)
\[ L = \{ww : w \in \{a, b\}^*\} \]
\[ a^m b^m a^m b^m = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1 \]

**Case 3:** \( v \) overlaps the first \( a^m b^m \)
\( y \) is in the first \( b^m \)

\[ a^m b^{k_2} a^{k_1} b^{k_3} a^m b^m = uv^2 xy^2 z \notin L \]

However, from Pumping Lemma: \( uv^2 xy^2 z \in L \)

**Contradiction!!!**
$$L = \{ww : w \in \{a, b\}^*\}$$

$$a^m b^m a^m b^m = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1$$

**Case 4:** $v$ in the first $a^m$

$y$ Overlaps the first $a^m b^m$

Analysis is similar to case 3
Other cases: \(vxy\) is within

\[
\begin{array}{c}
a^m b^m a^m b^m \\
\text{or} \\
a^m b^m a^m b^m \\
\text{or} \\
a^m b^m a^m b^m
\end{array}
\]

Analysis is similar to case 1:
More cases: $vxy$ overlaps \[ a^m b^m a^m b^m \]

or

\[ a^m b^m a^m b^m \]

Analysis is similar to cases 2,3,4:
There are no other cases to consider

Since $|vxy| \leq m$, it is impossible for $vxy$ to overlap:

\[ a^m b^m a^m b^m \]

nor

\[ a^m b^m a^m b^m \]

nor

\[ a^m b^m a^m b^m \]
In all cases we obtained a contradiction

Therefore: The original assumption that \[ L = \{ ww : w \in \{a,b\}^* \} \]
is context-free must be wrong

Conclusion: \[ L \text{ is not context-free} \]
Non-context free languages

\{a^n b^n c^n : n \geq 0\} \quad \{ww : w \in \{a,b\}\}

\{a^n! : n \geq 0\}

Context-free languages

\{a^n b^n : n \geq 0\} \quad \{ww^R : w \in \{a,b\}^*\}
Theorem: The language

\[ L = \{a^n! : n \geq 0\} \]

is \textbf{not} context free

Proof: Use the Pumping Lemma for context-free languages
\[ L = \{a^n!: n \geq 0\} \]

Assume for contradiction that \( L \) is context-free

Since \( L \) is context-free and infinite we can apply the pumping lemma
\[ L = \{ a^n! : n \geq 0 \} \]

Pumping Lemma gives a magic number \( m \) such that:

Pick any string of \( L \) with length at least \( m \), we pick: \( a^m! \in L \)
\[ L = \{ a^n! : n \geq 0 \} \]

We can write: \[ a^m! = uvxyz \]

with lengths \( |vxy| \leq m \) and \( |vy| \geq 1 \)

Pumping Lemma says:

\[ uv^i xy^i z \in L \quad \text{for all} \quad i \geq 0 \]
\[ L = \{ a^n! : n \geq 0 \} \]

\[ a^m! = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1 \]

We examine all the possible locations of string \( vxy \) in \( a^m! \)

How many cases to consider?
\[ L = \{a^n! : n \geq 0\} \]

\[ a^m! = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1 \]

We examine **all** the possible locations of string \( vxy \) in \( a^m! \)

There is only one case to consider
\[ L = \{ a^n! : n \geq 0 \} \]

\[ a^m! = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1 \]

\[ v = a^{k_1} \quad y = a^{k_2} \quad 1 \leq k_1 + k_2 \leq m \]
\[ L = \{a^n! : n \geq 0\} \]
\[ a^m! = uvxyz \quad \mid vxy \mid \leq m \quad \mid vy \mid \geq 1 \]

\[ m! + k_1 + k_2 \]

\[ u \quad v^2 \quad x \quad y^2 \quad z \]

\[ v = a^{k_1} \quad y = a^{k_2} \quad 1 \leq k_1 + k_2 \leq m \]
\[ L = \{a^{n!} : n \geq 0\} \]

\[ a^m! = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1 \]

\[ a \underbrace{..........}_m+k z \]

\[ k = k_1 + k_2 \]

\[ v = a^{k_1} \quad y = a^{k_2} \quad 1 \leq k \leq m \]
\[ L = \{a^n! : n \geq 0\} \]

\[ a^{m!} = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1 \]

\[ a^{m!+k} = uv^2xy^2z \]

\[ 1 \leq k \leq m \]
\[ L = \{a^n! : n \geq 0\} \]

\[ a^{m!+k} = uv^2x y^2z \quad 1 \leq k \leq m \]

Is \( a^{m!+k} \in L \)?
Since $1 \leq k \leq m$, for $m \geq 2$ we have:

$$m! + k \leq m! + m$$

$$< m! + m! m$$

$$= m!(1 + m)$$

$$= (m + 1)!$$

$$m! < m! + k < (m + 1)!$$
\[ L = \{a^n! : n \geq 0 \} \]

\[ a^m! = uvxyz \quad \mid vxy \mid \leq m \quad \mid vy \mid \geq 1 \]

\[ m! < m! + k < (m + 1)! \]

\[ a^{m! + k} = uv^2xy^2z \not\in L \]
\[ L = \{ a^n! : n \geq 0 \} \]

\[ a^m! = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1 \]

However, from Pumping Lemma: \[ uv^2xy^2z \in L \]

\[ a^{m!+k} = uv^2xy^2z \notin L \]

**Contradiction!!**
We obtained a contradiction

Therefore: The original assumption that

$$L = \{a^n! : n \geq 0\}$$

is context-free must be wrong

Conclusion: $L$ is not context-free
Non-context free languages

\{a^n b^n c^n : n \geq 0\} \quad \{w w : w \in \{a, b\}\}

\{a^{n^2} b^n : n \geq 0\} \quad \{a^n ! : n \geq 0\}

Context-free languages

\{a^n b^n : n \geq 0\} \quad \{w w^R : w \in \{a, b\}^*\}
Theorem: The language
\[ L = \{a^{n^2}b^n : n \geq 0\} \]
is not context free

Proof: Use the Pumping Lemma for context-free languages
\[ L = \{ a^{n^2} b^n : n \geq 0 \} \]

Assume for contradiction that \( L \) is context-free

Since \( L \) is context-free and infinite we can apply the pumping lemma
\[ L = \{ a^{n^2} b^n : n \geq 0 \} \]

Pumping Lemma gives a magic number \( m \) such that:

Pick any string of \( L \) with length at least \( m \) we pick: \( a^{m^2} b^m \in L \)
\[ L = \{ a^{n^2} b^n : n \geq 0 \} \]

We can write:

\[ a^m b^m = uvxyz \]

with lengths \(|vxy| \leq m\) and \(|vy| \geq 1\)

Pumping Lemma says:

\[ uv^i xy^i z \in L \quad \text{for all} \quad i \geq 0 \]
\[ L = \{a^{n^2}b^n : n \geq 0\} \]

\[ a^{m^2}b^m = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1 \]

We examine **all** the possible locations

of string \( vxy \) in \( a^{m^2}b^m \)
$L = \{a^{n^2}b^n : n \geq 0\}$

$a^m b^m = uvxyz$ \quad |vy| \leq m \quad |vy| \geq 1

**Most complicated case:**

\begin{itemize}
  \item $v$ is in $a^m$
  \item $y$ is in $b^m$
\end{itemize}

\begin{center}
\begin{tikzpicture}
  \node (a) at (0,0) {$a$};
  \node (b) at (1.5,0) {$b$};
  \node (u) at (0,-1) {$u$};
  \node (v) at (0.5,-1) {$v$};
  \node (x) at (1.0,-1) {$x$};
  \node (y) at (1.5,-1) {$y$};
  \node (z) at (2.0,-1) {$z$};
  \draw (a) -- (b);
  \draw (u) -- (v);
  \draw (x) -- (y);
  \draw (y) -- (z);
  \node at (0.75,0.5) {$m^2$};
  \node at (1.5,0.5) {$m$};
\end{tikzpicture}
\end{center}
\[ L = \{a^{n^2}b^n : n \geq 0\} \]

\[ a^m b^m = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1 \]

\[ v = a^{k_1} \quad y = b^{k_2} \quad 1 \leq k_1 + k_2 \leq m \]
\[ L = \{ a^{n^2} b^n : n \geq 0 \} \]
\[ a^m b^n = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1 \]

Most complicated sub-case: \( k_1 \neq 0 \) and \( k_2 \neq 0 \)

\[ v = a^{k_1} \quad y = b^{k_2} \quad 1 \leq k_1 + k_2 \leq m \]
\[ L = \{a^{n^2} b^n : n \geq 0\} \]

\[ a^m b^m = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1 \]

Most complicated sub-case: \( k_1 \neq 0 \) and \( k_2 \neq 0 \)

\[ v = a^{k_1} \quad y = b^{k_2} \quad 1 \leq k_1 + k_2 \leq m \]

\[ \begin{array}{c}
\text{a} & \ldots & \text{a} & \text{b} & \ldots & \text{b} \\
\text{u} & \text{v}_0 & \text{x} & \text{y}_0 & \text{z}
\end{array} \]
\[ L = \{a^{n^2} b^n : n \geq 0\} \]

\[ a^m b^m = u v x y z \quad \mid v x y \mid \leq m \quad \mid v y \mid \geq 1 \]

**Most complicated sub-case:** \( k_1 \neq 0 \) and \( k_2 \neq 0 \)

\[ v = a^{k_1} \quad y = b^{k_2} \quad 1 \leq k_1 + k_2 \leq m \]

\[ a^{m^2 - k_1} b^{m - k_2} = u v^0 x y^0 z \]
\[ a^{m^2-k_1} b^{m-k_2} \]

\[ k_1 \neq 0 \text{ and } k_2 \neq 0 \quad 1 \leq k_1 + k_2 \leq m \]

\[ (m-k_2)^2 \leq (m-1)^2 \]

\[ = m^2 - 2m + 1 \]

\[ < m^2 - k_1 \]

\[ m^2 - k_1 \neq (m-k_2)^2 \]
\[ L = \{ a^{n^2} b^n : n \geq 0 \} \]

\[ a^{m^2} b^m = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1 \]

\[ m^2 - k_1 \neq (m - k_2)^2 \]

\[ a^{m^2-k_1} b^{m-k_2} = uv^0 x y^0 z \notin L \]
\[ L = \{ a^{n^2} b^n : n \geq 0 \} \]

\[ a^{m^2} b^m = uvxyz \quad |vxy| \leq m \quad |vy| \geq 1 \]

However, from Pumping Lemma: \( uv^0 xy^0 z \in L \)

\[ a^{m^2-k_1} b^{m-k_2} = uv^0 xy^0 z \notin L \]

Contradiction!!!
When we examine the rest of the cases we also obtain a contradiction
In all cases we obtained a contradiction

Therefore: The original assumption that

\[ L = \{a^{n^2}b^n : n \geq 0\} \]

is context-free must be wrong

**Conclusion:** \( L \) is not context-free