1. Consider the function \( Q_1(x, n) = x^n \), where \( n \) is a positive integer and \( 0 \leq x \leq 1 \) is a real number.
   
a. Sketch the function \( Q_1(x, n) \) inside the unit square (with opposite corners at the points \((0, 0)\) and \((1, 1)\)). Show that the area under the curve is exactly \( 1/(n+1) \). [Hint: Use calculus!]

b. Explain how to use Monte Carlo simulation to find the approximate value of \( 1/5 \) with a program that uses the function \( Q_1(x, n) \) and repeated calls to a random number generator. [Hint: Writing the entire algorithm in pseudocode is the easiest way to cover all the details.]

2. Continuing with the example from question 1, let’s suppose you wrote the complete Monte Carlo simulation program, described above, and then ran it six times to generate six independent estimates for the value of \( 1/5 \). In each of those six runs, the program uses the random number generator to generate a total of 100 sample points on the unit square, and then counts how many of those sample points satisfy the condition for estimating \( 1/5 \) from the function \( Q_1(x, n) \). (Yes, I know this is vague, but I’m not going to give away the answer to the previous question!) At the end of those six runs, you are left with the following set of data:

   \[
   20 \quad 26 \quad 23 \quad 30 \quad 22 \quad 29
   \]

a. What is the correct value of \( n \) for estimating \( 1/5 \) from the function \( Q_1(x, n) \)?
b. Since your program generates each of the six data points by combining the results from many independent calculations, we know those points can be modeled by a specific probability distribution function. What is the name of that distribution, and what is the justification for choosing it?

c. In theory, if the program was working correctly for estimating \(1/5\), what should be the average count from a group of 100 sample points that satisfy the condition from the function \(Q1(x, n)\)?

d. Find the sample mean and sample variance for the above data set.

e. Construct a 95% confidence interval for the average count produced by your program, based on the above data you collected from 6 replications. [HINT: Since you shouldn’t be using a calculator, you may assume that \(\sqrt{6} \approx 2.5\). Does your confidence interval include your theoretical result from part c? What does that tell you about the correctness of your program?]
3. This problem develops a CSIM model for customers going to see a new hit documentary movie about gaining confidence called “The Same Day Again and Again in the Life of a CS177 Student.” Assume the box office opens at time zero, and sells tickets for hourly showings starting at times 1, 2, 3, \ldots.

**Description of the box office.** Since the capacity of the theatre auditorium is 100 customers, the box office has exactly 100 tickets available for each showing. At time zero, the box office starts selling tickets for show\_time = 1. When clock = show\_time (or when all 100 tickets for show\_time have been sold — whichever occurs first), the box office destroys any unsold tickets for show\_time, increments show\_time, and begins selling tickets for this later showing. Notice that the box office always has tickets available, but customers might need to wait several hours for their showing to begin.

**Description of the customers.** Customers visit the box office one-at-a-time in first-come first-served order, to purchase a single ticket for the next available showing (i.e., not necessarily the next showing!). Thereafter, s/he waits in the theatre lobby (drinking soda and playing video games) until the show\_time listed on the ticket, then enters the auditorium and stays there until the end of the movie.

a. Suppose we create one (permanent) CSIM process for the box office, a separate process for each customer, and let them interact through an event-based dialog similar to the airport shuttle. Briefly explain why customers must reserve a facility before engaging in the ticket-buying dialog.

b. Let’s define one event ready2buy to allow the customer to ask the box office to issue a ticket, and another event hereUgo for the box office to tell the customer to take it and leave. Write the CSIM declarations for these two events.

c. Assume the dialog is based on the global variable show\_time which is updated by the box office before setting the hereUgo event, and then read by the customer before it departs. Write the CSIM code fragments that belong to the box office and customer processes to carry out this ticket-issue dialog.
d. Since the number of showings is (potentially) unlimited, we cannot create an eventset with a separate element for every individual showing. Thus the customer process cannot simply wait for a unique show_time-specific event to occur before entering the auditorium. Explain how to write the customer process if the movie theatre uses a single event called newShowStarting to tell its customers that the next show (i.e., not necessarily their show) is now ready to begin. Your code must ensure that the customer does not try to enter the auditorium until the start of the correct showing.

e. We really don’t need to make the box office a separate process if we can trust the customers to be honest (i.e., leave their money in exchange for a ticket from the next available showing). Replace your code fragments from part (c) by a single code fragment belonging to the customer process that carries out the entire ticket purchase dialog without a separate box office process and without needing any events. [Hint: You still need the facility.]

Formulas:

\[
\bar{X}(n) = \frac{1}{n} \sum_{i=1}^{n} X_i \\
S^2(n) = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X}(n))^2
\]

\[
Prob[abs(\bar{X}(n) - \mu)\sqrt{\frac{\sigma^2}{n}} < Z_{1-\alpha/2}] = 1 - \alpha \\
Prob[abs(\bar{X}(n) - \mu)\sqrt{\frac{S^2(n)}{n}} < t_{(n-1)1-\alpha/2}] = 1 - \alpha
\]