Quiz 1 Solutions (version A)

Solution 1:

(a) The power set of $Y$ is $\mathcal{P}(Y) = \{\emptyset, \{a\}, \{e\}, \{h\}, \{a, e\}, \{a, h\}, \{e, h\}, \{a, e, h\}\}$

(b) The union of $X$ and $Y$ is $X \cap Y = \{a, b, c, d, e, f, h\}$

(c) The number of four-element subsets of $X$ is $\binom{6}{4} = 15$

(d) The number of ways to order all elements of $X$ is $6! = 720$

(e) The number of functions that map $Y$ into $X$ is $6^3 = 216$

Solution 2:

<table>
<thead>
<tr>
<th>statement</th>
<th>T/F</th>
<th>justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\exists x \in \mathbb{Z} : x^3 + 4x^2 - 2x + 3 = 0$</td>
<td>F</td>
<td>candidate roots are divisors of 3: 1,-1,3,-3, and none of them satisfies the equation</td>
</tr>
<tr>
<td>$\exists x \in \mathbb{R} : x^2 + 3x + 3 = 0$</td>
<td>F</td>
<td>discriminant is $3^2 - 4 \cdot 3 = -3$ it’s negative, so there is no solution in real numbers</td>
</tr>
<tr>
<td>$\forall x \in \mathbb{Z} : (-2)^{2x} &gt; 0$</td>
<td>T</td>
<td>$(-2)^{2x} = ((-2)^2)^x = 4^x &gt; 0$</td>
</tr>
<tr>
<td>$\forall x \in \mathbb{R} \exists y \in \mathbb{R} : 2x^2 = y^2 + 4$</td>
<td>F</td>
<td>for $x = 0$ we get $0 = y^2 + 4$, which does not have a solution in real numbers</td>
</tr>
<tr>
<td>$\exists x \in \mathbb{R} \forall y \in \mathbb{R} : x^2y - 3y = 0$</td>
<td>T</td>
<td>taking $x = \sqrt{3}$, equation becomes identity, no matter what $y$ is</td>
</tr>
</tbody>
</table>

Solution 3: We first verify the base case, for $n = 0$. If $n = 0$ then the left-hand side is 0 and the right-hand side is $(0 + 1)! - 1 = 0$ as well. So the equation is true in the base case.
In the inductive step, assume the identity holds for some \( n \), that is \( \sum_{i=1}^{n} i \cdot i! = (n+1)! - 1 \). We now want to prove that it also holds for the next integer \( n + 1 \), that is \( \sum_{i=1}^{n+1} i \cdot i! = (n+2)! - 1 \).

We start with the left-hand side and proceed as follows:

\[
\sum_{i=1}^{n+1} i \cdot i! = \sum_{i=1}^{n} i \cdot i! + (n + 1) \cdot (n + 1)!
\]

\[
= (n + 1)! - 1 + (n + 1) \cdot (n + 1)!
\]

apply the inductive assumption

\[
= (n + 1)! \cdot (1 + (n + 1)) - 1
\]

factor out \( (n + 1)! \)

\[
= (n + 1)! (n + 2) - 1 = (n + 2)! - 1,
\]

algebra

which gives us the desired equality for \( n + 1 \).

Summarizing, the identity holds for \( n = 0 \), and we proved that if it holds for some \( n \) then it holds for \( n + 1 \) as well. Thus the identity holds for all \( n \geq 0 \).