CS/MATH 111 SPRING 2011
Final Test

- The test is 2 hours and 30 minutes long, starting at 8:10AM and ending at 10:40AM
- There are 8 problems on the test. Each problem is worth 10 points.
- Write legibly. What can’t be read won’t be credited.
- Before you start:
  - Make sure that your final has all 8 problems
  - Put your name and your student ID on each page
Problem 1: (a) State the Master Theorem for solving divide-and-conquer recurrence equations.

(b) Give (asymptotic) solution to the recurrence equations below.

<table>
<thead>
<tr>
<th>Recurrence Equation</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T(n) = 9T(n/3) + 2n^3$</td>
<td>$O(n^3)$</td>
</tr>
<tr>
<td>$T(n) = 3T(n/4) + 5n$</td>
<td>$O(n^2)$</td>
</tr>
<tr>
<td>$T(n) = 4T(n/2) + 3n^2$</td>
<td>$O(n^2)$</td>
</tr>
<tr>
<td>$T(n) = 4T(n/2) + 3$</td>
<td>$O(n^2)$</td>
</tr>
<tr>
<td>$T(n) = T(n/3) + 5$</td>
<td>$O(n^3)$</td>
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</table>
**Problem 2:** Prove that $1 + 3 + 5 + \ldots + (2n - 1) = n^2$ for any integer $n \geq 1$. (The expression on the left-hand-side is the sum of the first $n$ odd natural numbers.) You can use mathematical induction or any other proof method.
Problem 3: In the RSA, suppose that Bob’s public key is $P_B = (91, 5)$. (a) Determine $d$, the secret exponent. Show your work. (b) Suppose that Bob receives ciphertext $C = 3$. Decrypt $C$. Show your work.
**Problem 4:** We have four types of blocks: $A$, $U$, $B$, $T$. Let $Q_n$ denote the number of different words of length $n$ that can be formed from these blocks. For example, for $n = 3$ there are 16 words:

- $AAA$, $AUA$, $AUA$, $AUU$, $UAA$, $UAU$, $UUA$, $UUU$
- $ABB$, $ATT$, $UBB$, $UTT$, $BBA$, $BBU$, $TTA$, $TTU$

so $Q_3 = 16$. Give a formula for $Q_n$. You must give a complete derivation: First give a recurrence equation and justify it, and then solve it. Show your work, all steps.
Problem 5: (a) Give a complete statement of Hall’s Theorem.

(b) Give a complete statement of Kuratowski’s Theorem.
Problem 6: Kevin is planning a 32-day trip to Scandinavia. He wants to spend at least 3 days in Finland, then between 7 and 14 days in Sweden, and later between 6 and 11 days in Norway. Compute the number of possible itineraries for his trip.
Problem 7: We are given four elements $a, b, c, d$ such that $a < b$ and $c < d$. Give a decision tree that sorts these elements and has depth at most 3. (You can think about this problem as merging two sorted pairs of elements into one sorted sequence.)
Problem 8: Recall that any planar graph with at least three vertices satisfies $m \leq 3n - 6$ (where $m, n$ denote the numbers of edges and vertices). Use this inequality to prove that each planar graph has a vertex of degree at most 5.