Dirac’s Theorem

Recall that a Hamiltonian cycle in a graph $G = (V, E)$ is a cycle that visits each vertex exactly once. Unlike for Euler cycles, no simple characterization of graphs with Hamiltonian cycles is known. In fact, the question whether a given graph has a Hamiltonian cycle is known to be $NP$-complete – a technical term that, for all practical purposes, implies that this question cannot be solved efficiently.

Some conditions that imply the existence of Hamiltonian cycles are known though. A rather obvious intuition is that if a graph is sufficiently dense (has enough edges) then it should have long cycles. The theorem below shows that this intuition is indeed right.

**Theorem 1** (Dirac’s theorem) Let $G = (V, E)$ be a graph with $n$ vertices in which each vertex has degree at least $n/2$. Then $G$ has a Hamiltonian cycle.

**Proof:** The proof is by an explicit construction, that is, we show that if $G$ satisfies the condition in the theorem that we can construct a Hamiltonian cycle in $G$.

The idea is to pick some vertex $v_1$ arbitrarily and gradually extend a path $P$ starting from $v_1$, say $P = v_1v_2...v_k$, where all vertices $v_j$ are different. Eventually, if $k = n$, $P$ will be a Hamiltonian path.

Initially, $P = (v_1)$. Suppose that we have already constructed $P = v_1v_2...v_k$. We now show that as long as $k < n$ we can always extend $P$.

If $v_k$ has a neighbor $u \in V$ that is not on $P$, then it is easy to extend $P$, for we can simply append $u$ at the end of $P$. In other words, we can take $v_{k+1} = u$ and the new extended path will be $v_1v_2...v_kv_{k+1}$.

The second case is when all neighbors of $v_k$ are on $P$. This case is a bit more tricky. The idea is this: We will show that there is a neighbor $v_j$ of $v_k$ such that $v_{j+1}$ has a neighbor outside $P$. Then we will perform a *switch operation* that transforms $P$ into the following path: $v_1v_2...v_jv_kv_{k-1}...v_{j+1}u$, as in the figure below:

![Diagram](https://via.placeholder.com/150)

Notice that the new path is indeed longer than $P$ by one vertex.

It is now sufficient to prove that such vertex $v_j$ always exists. Since all neighbors of $v_k$ are on $P$ and are different than $v_k$, we have $k - 1 \geq \deg(v_k) \geq n/2$, so $k \geq n/2 + 1$. Let’s do this: for each neighbor $v_j$ of $v_k$, we mark the next vertex on $P$, that is $v_{j+1}$. Since *all* neighbors of $v_k$ are on $P$, this way we will mark $\deg(v_k)$ vertices.
Consider any vertex \( u \) not on \( P \). If none of \( u \)'s neighbors were marked, then by adding the numbers of \( u \)'s neighbors, the marked vertices, and \( u \) itself, we would get that the total number of vertices in \( G \) is at least \( \deg(u) + \deg(v_k) + 1 \geq n/2 + n/2 + 1 > n \) – a contradiction. Therefore there must be a marked vertex that is a neighbor of \( u \). But this means, exactly, that there will be a vertex \( v_j \) as in the figure above, and the switch operation can be applied.

Summarizing what we’ve done so far, the above argument shows that \( G \) has a Hamiltonian path \( P = v_1v_2...v_n \). But the theorem actually says that \( G \) has a Hamiltonian cycle, so we are not really done yet. This is left as an exercise (for extra credit!!!) In other words, you need to show how (under thee assumptions from the theorem) you can convert \( P \) into a Hamiltonian cycle. \( \square \)