Quiz 1 Solutions (version A)

Solution 1:

(a) The power set of Y is $\mathcal{P}(Y) = \{\emptyset, \{a\}, \{e\}, \{h\}, \{a,e\}, \{a,h\}, \{e,h\}, \{a,e,h\}\}$

(b) The union of X and Y is $X \cap Y = \{a,b,c,d,e,f,h\}$

(c) The number of four-element subsets of X is $\binom{6}{4} = 15$

(d) The number of ways to order all elements of X is $6! = 720$

(e) The number of functions that map Y into X is $6^3 = 216$

Solution 2:

statement	T/F	justification
$\exists\, x \in \mathbb{Z} \,:\, x^3 + 4x^2 - 2x + 3 = 0$	F	candidate roots are divisors of 3: 1,-1,3,-3, and none of them satisfies the equation
$\exists\, x \in \mathbb{R} \,:\, x^2 + 3x + 3 = 0$	F	discriminant is $3^2 - 4 \cdot 3 = -3$ it's negative, so there is no solution in real numbers
$\forall\, x \in \mathbb{Z} \,:\, (-2)^{2x} > 0$	T	$(-2)^{2x} = ((-2)^2)^x = 4^x > 0$
$\forall\, x \in \mathbb{R}\; \exists\, y \in \mathbb{R} \,:\, 2x^2 = y^2 + 4$	F	for $x = 0$ we get $0 = y^2 + 4$, which does not have a solution in real numbers
$\exists\, x \in \mathbb{R}\; \forall\, y \in \mathbb{R} \,:\, x^2 y - 3y = 0$	T	taking $x = \sqrt{3}$, equation becomes identity, no matter what y is

---

Solution 3: We first verify the base case, for $n = 0$. If $n = 0$ then the left-hand side is 0 and the right-hand side is $(0+1)! - 1 = 0$ as well. So the equation is true in the base case.

In the inductive step, assume the identity holds for some n, that is $\sum_{i=1}^{n} i \cdot i! = (n+1)! - 1$. We now want to prove that it also holds for the next integer $n+1$, that is $\sum_{i=1}^{n+1} i \cdot i! = (n+2)! - 1$.

We start with the left-hand side and proceed as follows:

$$
\begin{aligned}
\sum_{i=1}^{n+1} i \cdot i! &= \sum_{i=1}^{n} i \cdot i! + (n+1) \cdot (n+1)! && \text{separate last term from the sum} \\
&= (n+1)! - 1 + (n+1) \cdot (n+1)! && \text{apply the inductive assumption} \\
&= (n+1)! \cdot [1 + (n+1)] - 1 && \text{factor out } (n+1)! \\
&= (n+1)!(n+2) - 1 = (n+2)! - 1, && \text{algebra}
\end{aligned}
$$

which gives us the desired equality for $n+1$.

Summarizing, the identity holds for $n = 0$, and we proved that if it holds for some n then it holds for $n+1$ as well. Thus the identity holds for all $n \geq 0$.
