Quiz 1 Solutions (version B)

Solution 1:

(a) The union of $U$ and $V$ is $U \cap V = \{a, b, c, d, e, f, h\}$

(b) The power set of $V$ is $\mathcal{P}(V) = \{\emptyset, \{a\}, \{e\}, \{h\}, \{a, e\}, \{a, h\}, \{e, h\}, \{a, e, h\}\}$

(a) The number of functions that map $V$ into $U$ is $6^3 = 216$

(c) The number of four-element subsets of $U$ is $\binom{6}{4} = 15$

(d) The number of ways to order all elements of $U$ is $6! = 720$

Solution 2:

<table>
<thead>
<tr>
<th>statement</th>
<th>T/F</th>
<th>justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\forall x \in \mathbb{Z} : (-3)^{2x} &gt; 0$</td>
<td>Y</td>
<td>$(-3)^{2x} = ((-3)^2)^x = 9^x &gt; 0$</td>
</tr>
<tr>
<td>$\exists x \in \mathbb{Z} : x^3 + 4x^2 - 2x + 3 = 0$</td>
<td>F</td>
<td>candidate roots are divisors of 3: 1,-1,3,-3, and none of them satisfies the equation</td>
</tr>
<tr>
<td>$\exists x \in \mathbb{R} : x^2 + 4x + 5 = 0$</td>
<td>F</td>
<td>discriminant is $4^2 - 4 \cdot 5 = -4$ it’s negative, so there is no solution in real numbers</td>
</tr>
<tr>
<td>$\exists x \in \mathbb{R} \forall y \in \mathbb{R} : x^2y - 5y = 0$</td>
<td>T</td>
<td>taking $x = \sqrt{5}$, equation becomes identity, no matter what $y$ is</td>
</tr>
<tr>
<td>$\forall x \in \mathbb{R} \exists y \in \mathbb{R} : 3x^2 = y^2 + 5$</td>
<td>F</td>
<td>for $x = 0$ we get $0 = y^2 + 5$, which does not have a solution in real numbers</td>
</tr>
</tbody>
</table>

Solution 3: We first verify the base case, for $m = 0$. If $m = 0$ then the left-hand side is 0 and the right-hand side is $(0 + 1)! - 1 = 0$ as well. So the equation is true in the base case.
In the inductive step, assume the identity holds for some $m$, that is $\sum_{j=1}^{m} j \cdot j! = (m+1)! - 1$. We now want to prove that it also holds for the next integer $m + 1$, that is $\sum_{j=1}^{m+1} j \cdot j! = (m + 2)! - 1$.

We start with the left-hand side and proceed as follows:

\[
\sum_{j=1}^{m+1} j \cdot j! = \sum_{j=1}^{m} j \cdot j! + (m + 1) \cdot (m + 1)! \quad \text{separate last term from the sum}
\]
\[
= (m + 1)! - 1 + (m + 1) \cdot (m + 1)! \quad \text{apply the inductive assumption}
\]
\[
= (m + 1)! \cdot [1 + (m + 1)] - 1 \quad \text{factor out } (m + 1)!
\]
\[
= (m + 1)!(m + 2) - 1 = (m + 2)! - 1, \quad \text{algebra}
\]

which gives us the desired equality for $m + 1$.

Summarizing, the identity holds for $m = 0$, and we proved that if it holds for some $m$ then it holds for $m + 1$ as well. Thus the identity holds for all $m \geq 0$. 

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