Problem 1: Amber needs to buy 31 bagels for a party. There are three flavors to choose from: poppyseed, blueberry, and garlic. She needs at least 3 poppyseed bagels, at most 13 blueberry bagels and at most 17 garlic bagels. How many possible combinations of bagels are there that satisfy these requirements? Show your work.

The problem is equivalent to computing the number of non-negative integer solutions to

\[ p + b + g = 34 \]
\[ p \geq 4 \]
\[ b \leq 13 \]
\[ g \leq 11 \]

After the substitution for \( p \), this reduces to computing the number of non-negative integer solutions to

\[ p + b + g = 30 \]
\[ b \leq 13 \]
\[ g \leq 11 \]

As in class, let \( S(P) \) be the number of solutions that satisfy condition \( P \). So we need to compute \( S(b \leq 13 \land g \leq 11) \). Denoting by \( S \) the number of all non-negative solutions, we have

\[ S(b \leq 13 \land g \leq 11) = S - S(b \geq 14 \lor g \geq 12) \]

We now compute \( S \):

\[ S = \binom{30 + 2}{2} = \binom{32}{2} = 496. \]

To compute \( S(b \geq 14 \lor g \geq 12) \), we use inclusion-exclusion:

\[ S(b \geq 14 \lor g \geq 12) = S(b \geq 14) + S(g \geq 12) - S(b \geq 14 \land g \geq 12) \]
\[ = \binom{30 - 14 + 2}{2} + \binom{30 - 12 + 2}{2} - \binom{30 - 26 + 2}{2} \]
\[ = \binom{18}{2} + \binom{20}{2} - \binom{6}{2} = 190 + 153 - 15 = 328. \]

So

\[ S(b \leq 11 \land g \leq 13) = 496 - 328 = 168. \]

\[ ^1 \text{You must use the method for counting integer partitions that we covered in class. Brute force listing of all solutions will not be credited.} \]
Problem 2: For each graph below determine the minimum number of colors necessary to color its vertices. Justify your answer, by giving a coloring and explaining why it is not possible to use fewer colors.

To give a coloring, use positive integers 1, 2, ... for colors and mark the color of each vertex in the box next to it. For ease of grading, assign color 1 to vertex $a$ and color 2 to vertex $b$.

Why the number of colors of $G$ is minimized?

$G$ requires 4 colors because it contains a 4-vertex clique consisting of vertices $a, b, d, g$.

Why the number of colors of $H$ is minimized?

$H$ requires 3 colors because it contains an odd-length cycle, for example $a, d, h, g, f, a$. 
Problem 3: (a) Compute $8^{-1} \pmod{19}$. Show your work.

Listing multiples of 19 plus 1, we get 20, 39, 58, 77, 96. Since $96 = 8 \cdot 12$, we have $8^{-1} \pmod{19} = 12$.

(b) Compute $5^{5983203} \pmod{101}$. Show your work.

Computing modulo 101, using Fermat’s theorem, we get

\[
5^{5983203} = 5^{59832 \cdot 100 + 3} \\
= (5^{100})^{59832} \cdot 5^3 \\
= 1 \cdot 125 = 24.
\]

(c) Compute $5^{17} \pmod{23}$. Show your work.

Computing modulo 23, we get

\[
9^{17} = 9 \cdot (9^2)^8 \\
= 9 \cdot 81^8 \\
= 9 \cdot 12^8 \\
= 9 \cdot 144^4 \\
= 9 \cdot 6^4 \\
= 9 \cdot 36^2 \\
= 9 \cdot 13^2 = 9 \cdot 169 = 9 \cdot 8 = 72 = 3.
\]
Problem 4: Solve the following recurrence equation:

\[ Y_n = 2Y_{n-1} + 3Y_{n-2} + 2^n \]
\[ Y_0 = 3 \]
\[ Y_1 = 4 \]

To find a particular solution, we try \( Y''_n = \beta 2^n \). After substituting, we get

\[ \beta 2^n = 2\beta 2^{n-1} + 3\beta 2^{n-2} + 2^n \]

which simplifies to

\[ 4\beta = 4\beta + 3\beta + 4 \]

so \( \beta = -\frac{4}{3} \). Thus \( Y''_n = -\frac{4}{3} 2^n \).

Next, we compute the general solution of the homogeneous equation. The characteristic equation is

\[ x^2 - 2x - 3 = 0 \]

The roots are \(-1, 3\). So the general solution for the homogeneous equation is

\[ Y'_n = \alpha_1 (-1)^n + \alpha_2 3^n. \]

We now combine it with the particular solution, getting the general solution of the inhomogeneous equation:

\[ Y_n = \alpha_1 (-1)^n + \alpha_2 3^n - \frac{4}{3} 2^n. \]

Plugging into the initial condition, we get equations:

\[ \alpha_1 + \alpha_2 - \frac{4}{3} = 3 \]
\[ -\alpha_1 + 3\alpha_2 - \frac{8}{3} = 4 \]

From these equations, \( \alpha_1 = \frac{19}{12} \) and \( \alpha_2 = \frac{11}{4} \). This gives us the final solution:

\[ Y_n = \frac{19}{12} (-1)^n + \frac{11}{4} 3^n - \frac{4}{3} 2^n. \]
Problem 5: For each integer $n \geq 1$ we define a tree $T_n$, as follows: $T_1$, $T_2$ and $T_3$ each consists of just a single node. For $n \geq 4$, $T_n$ is formed by creating four new nodes and attaching to them three copies of subtree $T_{\lfloor n/4 \rfloor}$, as in the picture below:

Let $R(n)$ be the number of nodes in $T_n$. For example, we have $R(1) = R(2) = R(3) = 1$, $R(4) = R(5) = \ldots = R(15) = 7$, and so on.

(a) Give a recurrence equation for $R(n)$ and justify it. (b) Then determine the asymptotic value of $R(n)$, expressing it using the $\Theta$-notation.

(Reminder: $\lfloor x \rfloor$ is the largest integer not larger than $x$. For example, $\lfloor 2.7 \rfloor = 2$ and $\lfloor 23/3 \rfloor = 7$.)

$T_n$ contains all nodes from the three copies of $T_{\lfloor n/4 \rfloor}$, plus 4 additional nodes. Therefore the number of nodes $R(n)$ satisfies the recurrence

$$R(n) = 3 \cdot R(\lfloor n/4 \rfloor) + 4.$$ 

To estimate $R(n)$, we use Master Theorem. We have $a = 3$, $b = 4$ and $d = 0$, so $a > b^d$. So the solution is

$$R(n) = \Theta(n^{\log_4 3}).$$
Problem 6: Consider numbers $A_n$ defined recursively as follows: $A_0 = A_1 = A_2 = 1$, and $A_n = A_{n-1} + A_{n-2} + A_{n-3}$ for all integers $n \geq 3$. Using mathematical induction, prove that $A_n \leq 2^n$ for all $n \geq 0$.

Base case: In the base case we verify that the inequality holds for $n = 0, 1, 2$. For $n = 0$, $A_0 = 1 \leq 2^0$, for $n = 1$, $A_1 = 1 \leq 2^1$, and for $n = 2$, $A_2 = 1 \leq 2^2$. So the inequality holds in the base case.

Inductive step: Now, let $k \geq 2$, and assume that $A_n \leq 2^n$ holds for all $n \leq k$. We show that it also holds for $k + 1$, that is $A_{k+1} \leq 2^{k+1}$. The derivation is as follows:

\[
A_{k+1} = A_k + A_{k-1} + A_{k-2} \\
\leq 2^k + 2^{k-1} + 2^{k-2} \quad \text{(from the inductive assumption)} \\
= 2^{k-2}(4 + 2 + 1) \\
= 7 \cdot 2^{k-2} \\
\leq 8 \cdot 2^{k-2} \\
= 2^{k+1}.
\]

This implies that $A_{k+1} \leq 2^{k+1}$, completing the proof.
Problem 7: Complete statements of the following theorems.

(d) Kuratowski’s Theorem: Let $G$ be a graph. $G$ is planar if and only if $G$ does not contain a subgraph that is a sub-division of $K_5$ or $K_{3,3}$.

(a) Euler’s Theorem: Let $G$ be a connected graph. $G$ has an Euler tour if and only if each vertex in $G$ has even degree.

(b) Dirac’s Theorem: Let $G$ be a graph with $n$ vertices. If each vertex in $G$ has degree at least $n/2$ then $G$ has a hamiltonian cycle.

(c) Hall’s Theorem: Let $G = (L, R, E)$ be a bipartite graph. $G$ has a perfect matching if and only if $|L| = |R|$ and for each $X \subseteq L$ we have $|N(X)| \geq |X|$.
Problem 8: Give the formulas for the following quantities. Provide a justification for each.

(a) (2 points) The number of all strings of length \( n \) formed from letters \( a, b, c, d, e, f \).

For each \( n \) positions we have 6 choices, so the number of strings is \( 6^n \).

(b) (2 points) The number of all strings of length \( n \) formed from letters \( a, b, c, d, e, f \) that contain exactly two \( a \)'s and exactly two \( b \)'s. (Here we assume \( n \geq 4 \)).

There are \( \binom{n}{2} \) choices for the positions that have \( a \)'s. Among the remaining \( n - 2 \) positions, there are \( \binom{n-2}{2} \) choices for the positions that have \( b \)'s. The remaining \( n - 4 \) positions can be filled in 4 ways each, for the total of \( 4^{n-4} \). So the answer is

\[
\binom{n}{2} \cdot \binom{n-2}{2} \cdot 4^{n-4} = \frac{1}{4} n(n-1)(n-2)(n-3)4^{n-4}.
\]

(c) (6 points) The number of all strings of length \( n \) formed from letters \( a, b, c, d, e, f \) that contain at least two \( a \)'s and at least two \( b \)'s. (Here we assume \( n \geq 4 \)).

Some notation will be helpful. Let \( \alpha \) be the number of \( a \)'s and \( \beta \) be the number of \( b \)'s in the string. Let also \( S(P) \) be the number of strings of length \( n \) with property \( P \). So we want to compute \( S(\alpha \geq 2 \land \beta \geq 2) \). Then

\[
S(\alpha \geq 2 \land \beta \geq 2) = 6^n - S(\alpha < 2 \lor \beta < 2)
\]

We now compute \( S(\alpha < 2 \lor \beta < 2) \), using the inclusion-exclusion principle and breaking into cases:

\[
S(\alpha < 2 \lor \beta < 2) = S(\alpha < 2) + S(\beta < 2) - S(\alpha < 2 \land \beta < 2)
= S(\alpha = 0) + S(\alpha = 1) + S(\beta = 0) + S(\beta = 1) - [ S(\alpha = 0 \land \beta = 0) + S(\alpha = 1 \land \beta = 0) + S(\alpha = 0 \land \beta = 1) + S(\alpha = 1 \land \beta = 1) ]
= 5^n + n5^{n-1} + 5^n + n5^{n-1} - [ 4^n + n4^{n-1} + n4^{n-1} + n(n-1)4^{n-2} ]
= (2n + 10)5^{n-1} - (n^2 + 7n + 16)4^{n-2}
\]

So the answer is

\[
S(\alpha \geq 2 \land \beta \geq 2) = 6^n - (2n + 10)5^{n-1} + (n^2 + 7n + 16)4^{n-2}
\]