**CS/MATH111 ASSIGNMENT 1 SOLUTIONS**

**Solution 1:** (a) The first nested for-loop prints \(3n^3\) words. We now determine the number of words printed by the second nested for-loop. The \(i\)-th iteration of the outer loop prints \(i^2\) words, and we need to add these values for \(i\) ranging from 1 to \(n\). Thus the total number of words printed can be written as

\[
f(n) = 3n^3 + \sum_{i=1}^{n} i^2\]

To find a formula for \(f(n)\), we use the formula for the sum of consecutive squares: \(\sum_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1)\). Plugging in, this will give us

\[
f(n) = 3n^3 + \frac{1}{6}n(n+1)(2n+1) = 3n^3 + \frac{1}{2}(2n^3 + 3n^2 + n) = \frac{10}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n\]

(b) Thus \(f(n)\) is a polynomial of degree 3, which implies that \(f(n) = \Theta(n^3)\).

**Solution 2:** (a) We will show that \(f(n) = \Theta(n^2)\).

For \(n \geq 1\) we have

\[
f(n) = 3n^2 + n + 5 \leq 3n^2 + n^2 + 5n^2 = 9n^2.
\]

So \(f(n) = O(n^2)\). Since \(n \geq 0\), we also have

\[
f(n) = 3n^2 + n + 5 \geq 3n^2,
\]

which implies that \(f(n) = \Omega(n^2)\). Putting these two bounds together, we obtain that \(f(n) = \Theta(n^2)\).

(b) We will show that \(g(n) = \Theta(3^n)\).

Since \((-2)^n \leq 2^n\), for all \(n \geq 0\) we have

\[
g(n) = 3^n + (-2)^n \leq 3^n + 2^n \leq 3^n + 3^n = 2 \cdot 3^n.
\]

So \(g(n) = O(3^n)\). To obtain a lower bound for \(g(n)\), note that \((-2)^n \geq -2^n\). Below we also show that \(2^n \leq \frac{1}{2}3^n\) for \(n \geq 2\). Thus

\[
g(n) = 3^n + (-2)^n \geq 3^n - 2^n \geq 3^n - \frac{1}{2}3^n = \frac{1}{2}3^n,
\]

which implies that \(g(n) = \Omega(3^n)\). Combined with the earlier upper bound, we obtain that \(g(n) = \Theta(3^n)\).

It remains to show that \(2^n \leq \frac{1}{2}3^n\) for \(n \geq 2\). Multiplying by \(2/2^n\), this inequality is equivalent to \((3/2)^n \geq 2\). The left-hand side is a geometric sequence that is increasing and it is larger than 2 when \(n = 2\). So it is also larger than 2 for all \(n \geq 2\). (The inequality \(2^n \leq \frac{1}{2}3^n\) for \(n \geq 2\) can also be proven by induction.)

**Solution 3:** (a) \(\frac{1}{2}n^5 + 98328n^2 + 3n = \Theta(n^5)\), because this is a polynomial of degree 8.

(b) \(n^5 \log n + 3\sqrt{n} + 5 = \Theta(n^5 \log n)\). Justification: This expression is trivially \(\Omega(n^5 \log n)\), because all terms are non-negative. It remains to show that it is also \(O(n^5 \log n)\). This is true because \(3\sqrt{n} + 5\) is a polynomial of degree \(1/2\), less than 5, so it’s dominated by the term \(n^5 \log n\).
(c) $n^5 / \log n + 3n^4 \log^4 n + 7n = \Theta(n^5 / \log n)$. Justification: This expression is trivially $\Omega(n^5 / \log n)$, because all terms are non-negative. It remains to show the matching upper bound. We have $3n^4 \log^4 n \leq n^5 / \log n$ for $n$ large enough, because this inequality is equivalent to $n \geq 3 \log^6 n$, which is true by the theorem covered in class. A similar argument shows that $7n \leq n^5 / \log n$ for $n$ large enough. So $n^5 / \log n + 3n^4 \log^4 n + 7n = O(n^5 / \log n)$.

(d) $5 \cdot 2^n + n^{12} \log n + 3 = \Theta(2^n)$. Justification: This expression is trivially $\Omega(2^n)$, because all terms are non-negative. It remains to show the matching upper bound. The inequality $3 \leq 2^n$ is trivial for $n \geq 2$. Regarding the term $n^{12} \log n$, using the theorems from class, $n^{12} \log n \leq n^{12} \cdot O(n) = O(n^{13}) = O(2^n)$. So the whole expression is also $O(2^n)$.

(e) $\log^2 n + n^{13} 2^n + 3^n = \Theta(3^n)$. Justification: This expression is trivially $\Omega(3^n)$, because all terms are non-negative. It remains to show the matching upper bound. By the theorems in class, $\log^2 n = O(n) = O(3^n)$. To estimate the second term, using theorems from class, $n^{13} 2^n = O(1.5^n) 2^n = O(3^n)$. So the whole expression is also $O(3^n)$.