UNIVERSITY OF CALIFORNIA, RIVERSIDE

DEPARTMENT OF COMPUTER SCIENCE

2006 DEPTH EXAMINATION IN THEORY OF OF COMPUTATION AND ALGORITHMS

- There are 10 problems on the test. Each problem is worth 10 points. The ordering of the problems is unrelated to their difficulty.
- Answer exactly **8** out of the 10 questions. Indicate below which two problems you do not want to have graded.
- Write legibly. What can't be read won't be credited.
- Algorithms can be described informally in pseudo-code.
- Good luck!

Name:

Skip Problems:

1.

2.

Theory Depth Exam 4/17/2006, 1:00PM-4:00PM, EBU2 202

Problem 1: Consider the following procedure which builds a heap when given as input an array A of n integers.

$\textsc{BuildHeap}(A : \textbf{array of int})$
for $i \leftarrow \lfloor n/2 \rfloor$ **downto** 1 **do**
    $\textsc{Heapify}(A, i)$

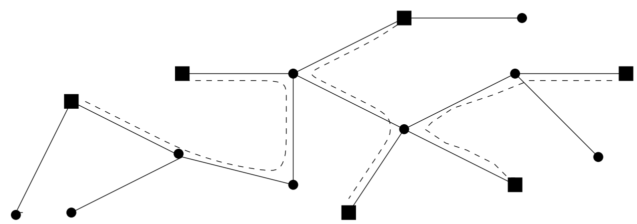
$\textsc{Heapify}(A : \textbf{array of int}, i : \textbf{int})$
$\mathit{left}, \mathit{right} \leftarrow 2i, 2i+1$
if $(\mathit{left} \leq n)$ **and** $(A[\mathit{left}] > A[i])$ **then** $\mathit{max} \leftarrow \mathit{left}$ **else** $\mathit{max} \leftarrow i$
if $(\mathit{right} \leq n)$ **and** $(A[\mathit{right}] > A[\mathit{max}])$ **then** $\mathit{max} \leftarrow \mathit{right}$
if $(\mathit{max} \neq i)$ **then**
    swap $A[i]$ with $A[\mathit{max}]$
    $\textsc{Heapify}(A, \mathit{max})$

$\textsc{BuildHeap}()$ starts with the element at position $\lfloor n/2 \rfloor$ and works backwards. $\textsc{Heapify}(A, i)$ pushes the element at position i down the heap as far as necessary. The elements in positions $\lfloor n/2 \rfloor + 1$ through n are leaves, and so they cannot be pushed further down. The other nodes are either in the correct position, or larger than their children; thus, a push-down fixes the sub-heap rooted at i without affecting the rest of the heap.

a. Express the time-complexity of BuildHeap as the sum of the number of operations performed at each level of the heap.

b. Show that the overall time complexity of the procedure BuildHeap is $O(n)$.

Problem 2: We are given a tree T with n nodes and a set X of distinct nodes in T of cardinality $2k$ (where $k \leq n/2$). A *matching* of X in T is a partition of X into k pairs of nodes (x_1, y_1), $(x_2, y_2), \ldots, (x_k, y_k)$. Given T and X we want to find a matching such that, for any two matched pairs (x_i, y_i) and (x_j, y_j), the simple paths p_i from x_i to y_i and p_j from x_j to y_j are edge disjoint. In the example illustrated below, the nodes in X are marked with a square and dashed lines show the paths between the matched vertices.



a. Prove that for each T and X there exists such a matching.

b. Give a linear-time algorithm to compute the matching and justify its correctness.

Problem 3: Consider the following REVERSE-DELETE algorithm for the Minimum Spanning Tree (MST) problem. Start with a connected edge-weighted graph $G = (V, E)$. Consider edges in order of decreasing cost (breaking ties arbitrarily). When considering an edge e, delete e from E unless doing so would disconnect the current graph. Prove that REVERSE-DELETE computes a MST.

Problem 4: Consider the following problem. Given a set L of n (assume n even) positive integers, partition the set into two subsets A and B each of size $n/2$, such that the absolute value of the difference between the sums of the integers in the two subsets is minimized. In other words, we want to minimize $|\sum_{x \in A} x - \sum_{y \in B} y|$ where $A \cup B = L$, $A \cap B = \emptyset$ and $|A| = |B| = n/2$. Give a pseudo-polynomial time algorithm for this problem and analyze its time complexity.

Problem 5: State and prove the max-flow/min-cut theorem (as presented in class).

Problem 6: Describe a sequence of languages $L_1, L_2, L_3, \ldots$ such that each L_n is recognized by an n-state deterministic finite automaton, but L_n is not recognized by any $(n-1)$-state deterministic finite automaton. Prove that your languages have the desired property.

Problem 7: Let M be a non-deterministic polynomial-time Turing machine. Define the *probability* that M accepts an input w to be the number of accepting computation paths, divided by the total number of computation paths, when M is run on w. Define $L_{1/2}(M)$ to be the set of words that are accepted by M with probability greater than 1/2.

a. Show that SAT equals $L_{1/2}(M)$ for some non-deterministic, polynomial-time Turing machine M.

b. Define the complexity class $C = \{L_{1/2}(M) : M \text{ is a non-deterministic poly-time TM}\}$. Show that if $C = \text{NP}$, then $\text{NP} = \text{co-NP}$.

Problem 8: Recall that the *configuration* of a Turing machine is defined by a triple (q, w, u), where q is the current state, $w \in \Sigma^*$ is the string to the left of the cursor, and $u \in \Sigma^*$ is the string to the right of cursor (including the symbol scanned by the cursor). Consider the RepeatsConfiguration problem defined below.

Instance: A description M of a Turing machine and input tape I;

Query: Does the execution of M on the input I ever repeat a configuration?

a. Is RepeatsConfiguration r.e.?

b. Is RepeatsConfiguration undecidable?

Justify your answers.

Problem 9: Define $\textsc{Clip}(L_1, L_2) = \{u : \exists w \in L_2 \text{ such that } uw \in L_1\}$. If L_1, L_2 are Turing-recognizable (r.e.) then so is $\textsc{Clip}(L_1, L_2)$? Justify your answer.

Problem 10: A *wheel* in a graph $G = (V, E)$ is a subgraph that consists of a cycle and a "center" node connected to all nodes in this cycle. Prove that finding the largest wheel in a graph G is NP-hard.