Space Complexity — Savitch's Theorem

Space Complexity of TMs

The **space complexity** of a deterministic Turing machine M

$\cdots$ length-wise, for each n, it is the maximum number of tape cells that M's head touches before it halts for any input of length n

The **space complexity** of a nondeterministic Turing machine M

$\cdots$ take the maximum for all computation paths of M on all inputs of length n

For any $f : \mathcal{N} \to \mathcal{N}$, a Turing machine M is $f(n)$ **space** if M's space complexity is f.

Note: Time bounded Turing machines always halt, but space bounded Turing machines may or may not halt.

Whether the input can be modified or not

For functions f such that $f(n) < n$ for some n, we treat the space complexity differently:

- the Turing machine has more than one tape,
- the input is given with **markers indicating two ends** and **will never be modified,**
- on each work tape $f(n)$ **may cells are provided with markers** indicating two ends.

Space Complexity Classes

Definition. *Let $f : \mathcal{N} \to \mathcal{N}$ be a function. Then*
$\mathrm{SPACE}(f(n)) = \{L \mid L$ *is decided by an $O(f(n))$ space deterministic Turing machine* $\}$, *and.*
$\mathrm{NSPACE}(f(n)) = \{L \mid L$ *is decided by an $O(f(n))$ space nondeterministic Turing machine* $\}$.

Example: $SAT \in (n)$.
Consider a 2-tape TM that, on input a formula ϕ of n variables, **tries all assignments to ϕ,** each viewed as an n bit binary number, and accepts if and only if one assignment that satisfies ϕ has been found.

Such a machine correctly decides SAT and $O(n)$ space.

Space Complexity May Depend on Encoding Schemes

Example: $CLIQUE \in \mathrm{SPACE}(\sqrt{n})$
Suppose that the adjacency matrix is used for representing a graph.

Consider a 2-tape TM that, on input a graph $G = (V, E)$ of some m modes and a number $k \geq 1$, **tries all possible subsets of V,** viewed as an m-bit binary number, and accepts if and only if one subset is of size k and is a clique in G.

Let the encoding length be n. Then $n \geq m^2$ and the space requirement is $O(m)$. So, $CLIQUE \in \mathrm{SPACE}(\sqrt{n})$.

Savitch's Theorem

Theorem. *Suppose $f(n) \geq C \log n$ for some constact $C > 0$. Then $\mathrm{NSPACE}(f(n)) \subseteq \mathrm{SPACE}(O(f(n)^2))$.*

Proof Let $f(n) \geq \log n$, $L \in \mathrm{NSPACE}(f(n))$ via a 2-tape NTM M with an unmodifiable input tape and a work tape. Let $x, |x| = n$, be an input. Since a marker is provided at each end of each tape, we can assume that M erases the work tape cells and moves the heads to the leftmost positions before halting so that **there is a unique accepting configuration.**

Encoding Configurations

Each configuration of M on x is a combination of the **state**, the **input head position**, the **work tape head position**, and the **work tape contents**, requiring $O(1)$ symbols, $O(\log n)$ symbols, $O(\log f(n))$ symbols, and $f(n)$ symbols, respectively.

Since $f(n) \geq C \log n$, the total symbols needed is at most $kf(n)$ for some fixed k.

There are at most $2^{lf(n)}$ configurations for some fixed l.

Reachability

For configurations c, c' and an integer $t \geq 0$, define
$CANYIELD(c, c', t) = 1$ **if c yields c' within 2^t steps and 0 otherwise.**

Let c_0 be the initial configuration and c_{accept} be the unique accepting configuration. Then $x \in L$ if and only if $CANYIELD(c_0, c_{accept}, lf(n)) = 1$.

For each c, c', $CANYIELD(c, c', 0) = 1$ if and only if $c \Rightarrow c'$, so whether $CANYIELD(c, c', 0) = 1$ can be tested by simulating all possible single step moved of the TM from configuration c.

For each c, c', and $t > 0$, $CANYIELD(c, c', t) = 1$ if and only if there is some d such that $CANYIELD(c, d, t-1) = CANYIELD(d, c', t-1) = 1$.

Deterministic Recursive Reachability Testing

Call $CANYIELD(c_0, c_{accept}, lf(n))$ and accept if and only if 1 is returned.

$CANYIELD(c, c't)$
if $t = 0$ **then** simulate all possible one-step moves of M
    from c, and return 1 if c' is generated
else
    for each configuration d
        if $CANYIELD(c, d, t-1) = CANYIELD(c, d, t-1) = 1$
            then return 1.
    return 0.

The recursion depth is t, so the space requirement is $O(kf(n) \cdot lf(n)) = O(f(n)^2)$. ∎