More NP-Completeness

Clique

We know: $\mathit{3SAT} \leq_P \mathit{CLIQUE}$, $\mathit{CLIQUE} \in \mathrm{NP}$, and $\mathit{3SAT}$ is NP-complete. So, CLIQUE is NP-complete.

Vertex Cover

A vertex cover of an undirected graph is a subset of nodes such that every edge touches a member of the subset.

$\mathit{VERTEX}\text{-}\mathit{COVER} = \{\langle G, k\rangle \mid G$ has a vertex cover of size $k\}$.

Theorem. *$\mathit{VERTEX}\text{-}\mathit{COVER}$ is NP-complete.*

Proof Proving $\mathit{VERTEX}\text{-}\mathit{COVER} \in \mathrm{NP}$ is easy. Guess a bit for each node to decide whether or not select the node. Check whether precisely k nodes are selected, if so, check whether the set of k nodes is a vertex cover.

Proof (cont'd)

Reduce $\mathit{3SAT}$ to $\mathit{VERTEX}\text{-}\mathit{COVER}$.

Let ϕ be an instance of $\mathit{3SAT}$ with n variables and m clauses. Define the graph G as follows:

- **the nodes:** **the literals** $v_i, \overline{v_i}, 1 \leq i \leq n$, and

  their occurrences $a_{i1}, a_{i2}, a_{i3} : 1 \leq i \leq m$,

  $\cdots$ a total of $3m + 2n$ nodes
- **the edges:** $(v_i, \overline{v_i})$, $1 \leq i \leq n$;

  $(a_{i1}, a_{i2}), (a_{i2}, a_{i3}), (a_{i3}, a_{i1})$, $1 \leq i \leq m$;

  for each i, $1 \leq i \leq n$, and j, $1 \leq j \leq 3$, connect a_{ij} and its corresponding literal.

Proof (cont'd)

We claim that G has an $(n + 2m)$ node vertex cover if and only if ϕ is in $\mathit{3SAT}$.

There are precisely n edges of the type $(v_i, \overline{v_i})$, $1 \leq i \leq n$. So an n-node vertex cover has to have at least one out of v_i and $\overline{v_i}$ for every i.

There are precisely m triangles for the clauses. At least two nodes have to be selected from each triangle to cover the triangle edges. So $2m$ nodes are needed.

Proof (cont'd)

A selection of $n + 2m$ nodes becomes a cover if and only if:

(*) **for each clause, the literal side of one "literal-occurrence" edge is selected, say l_k of (a_{ij}, l_k), where l_k is a literal, and two nodes $a_{ij'}$ and $a_{ij''}$ are selected, where $\{j, j', j''\} = \{1, 2, 3\}$**, and

(**) **for each variable precisely one of v_i and $\overline{v_i}$ is selected.**

So an $n + 2m$ node vertex cover corresponds to an assignment that satisfies each clause. ∎

Subset-Sum is NP-complete

$SUBSET\text{-}SUM$ is the problem of, given a multiset of numbers $z_1, \ldots, z_m$ and a number S, whether there is subset $y_1, \ldots, y_t$ of z_i's such that $y_1 + \cdots + y_t = S$.

Theorem. *$SUBSET\text{-}SUM$ is NP-complete.*

Proof Reduce $3SAT$ to $SUBSET\text{-}SUM$. The construction is reminiscent of the reduction from $3SAT$ to $VERTEX\text{-}COVER$, where the reduction generates a graph whose $n + 2m$ node cover has a property that at least one "literal-occurrence" edge of each triangle is touched and the rest of the nodes in each triangle is touched.

Proof (cont'd)

Let ϕ be a formula of n variables and m clauses. Introduce decimal numbers $y_1, \ldots, y_n$, $z_1, \ldots, z_n$, $c_1, \ldots, c_m$, $d_1, \ldots, d_m$, each of at most $n + m$ digits.

y_i: y_i has a 1 at the $(m+i)$th digit and has a 1 at position j if x_i appears in the jth clause; all the other positions have a 0

z_i: z_i has a 1 at the $(m+i)$th digit and has a 1 at position j if $\overline{x_i}$ appears in the jth clause; all the other positions have a 0

c_i, d_i: c_i has a 1 only at the ith position, d_i has a 1 only at the ith position,

S: S is the number that has a 3 at every position between 1 and m and has a 1 at every position between $m + 1$ and $m + n$

Proof (cont'd)

In order to generate S, exactly one of y_i and z_i has to be selected for every i so that the selection as a whole touches each bit position between 1 and m at least once (and at most three times). Such a selection is a satisfying assignment of ϕ. ∎