Pushdown Automata

Pushdown Automata

A **pushdown automaton** is a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_0, F)$ where Q, Σ, Γ, F are finite sets, and

1. Q is the **set of states**,
2. Σ is the **input alphabet**,
3. Γ is the **stack alphabet**,
4. $\delta : Q \times \Sigma_\epsilon \times \Gamma_\epsilon \to \mathcal{P}(Q \times \Gamma_\epsilon)$ is the **transition function**,
5. $q_0 \in Q$ is the start state, and
6. $F \subseteq Q$ is the **set of accept states**.

Pushdown Automata (cont'd)

At each computational step, a PDA does the following:

1. nondeterministically **decides whether to read the next input symbol and the current stack symbol**; for input symbol w_i and stack symbol s_j the choices are $(w_i, s_j), (\epsilon, s_j), (w_i, \epsilon), (\epsilon, \epsilon)$;
   (if the input or stack are empty, there are fewer than 4 choices);
2. if the choice is (u, v), then if $u \neq \epsilon$ it **reads (moves past)** u and if $v \neq \epsilon$ it **pops** v **from the stack**; and
3. it nondeterministically **selects the next state and a symbol to be put on the stack** according to the transition function.

It halts when **either (i)** there is **no next move**, **or (ii) no input symbols are left** and **the current state is an accept state.**

Pushdown Automata (cont'd)

$M = (Q, \Sigma, \Gamma, \delta, q_0, F)$ **accepts** a word $w \in \Sigma^*$ if w can be represented as $w_1 \cdots w_m$ with $w_1, \ldots, w_m \in \Sigma_\epsilon$ and there exist $r_0, \ldots, r_m \in Q$ and $s_0, \ldots, s_m \in \Gamma^*$ satisfying the following conditions:

1. $r_0 = q_0$ and $s_0 = \epsilon$.
2. For every i, $0 \leq i \leq m-1$, $(r_{i+1}, b) \in \delta(r_i, w_i, a)$, where $s_{i-1} = at$ and $s_i = bt$ for some $a, b \in \Gamma_\epsilon$ and $t \in \Gamma^*$.
3. $r_m \in F$.

Here the stack is read from top to bottom (or left to right).

PDA, Example 1

$L = \{0^n 1^n \mid n \geq 0\}$.

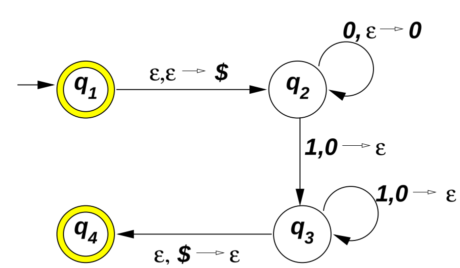
Design Idea

- Use a special symbol $ to **mark the bottom of the stack.** (Using ϵ wouldn't work because ϵ is *always* "there".)
- First put onto the stack all the **0s preceding the 1s.**
- Then try to **match the stacked 0s with the 1s.**
- The input string is in L if, and only if, **the bottom $\$$ is the top symbol when all the input symbols have been read.**

Example 1 (cont'd)

$\Gamma = \{0, 1, \$\}$, $Q = \{q_1, q_2, q_3, q_4\}$, $F = \{q_1, q_4\}$, q_1 is the initial state

We use q_1 to place the $ on the stack, q_2 to read 0s and push them onto the stack (and react to the first 1), and q_3 to read 1s and pop 0s from the stack (and react to the $ by going to accept state q_4). As a diagram:



Example 1 (cont'd)

As a transition table (where blank means empty set):

Input:	0			1			ϵ		
Stack:	0	$	ϵ	0	$	ϵ	0	$	ϵ
q_1									$\{(q_2, \$)\}$
q_2			$\{(q_2, 0)\}$	$\{(q_3, \epsilon)\}$					
q_3				$\{(q_3, \epsilon)\}$				$\{(q_4, \epsilon)\}$	

Let $(q, u, v), q \in Q, u \in \Sigma^*, v \in \Gamma^*$ denote the configuration in which **the state is q, the remaining input is u, and the stack word is v.**

000111 is accepted by a path: $(q_1, 000111, \epsilon) \Rightarrow (q_2, 000111, \$) \Rightarrow (q_2, 00111, 0\$) \Rightarrow (q_2, 0111, 00\$) \Rightarrow (q_2, 111, 000\$) \Rightarrow (q_3, 11, 00\$) \Rightarrow (q_3, 1, 0\$) \Rightarrow (q_3, \epsilon, \$) \Rightarrow (q_4, \epsilon, \epsilon)$.

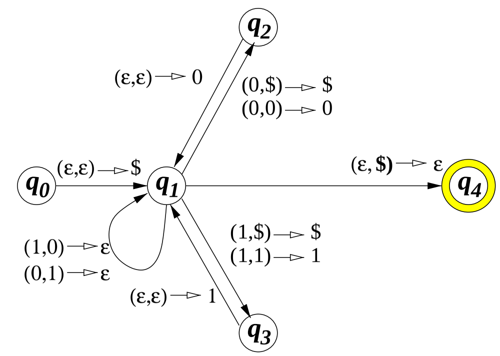
PDA, Example 2

$L = \{u \in \{0, 1\}^* \mid u$ has an equal number of 0s and 1s $\}$.
$Q = \{q_0, q_1, q_2, q_3, q_4\}$, $F = \{q_1\}$, $\Gamma = \{0, 1, \$\}$, q_0 is the initial state

	0			1			ϵ	
	0	1	$	0	1	$	ϵ	$
q_0							$(q_1, \$)$	
q_1	$(q_2, 0)$	(q_1, ϵ)	$(q_2, \$)$	(q_1, ϵ)	$(q_3, 1)$	$(q_3, \$)$		(q_4, ϵ)
q_2							$(q_1, 0)$	
q_3							$(q_1, 1)$	

Here $\{$ and $\}$ are omitted. State q_0 is used to place $ on the stack, q_1 is used to pop stack symbols "matching" a complementary input symbol (and to recognize when we're done), q_2 is used to push unmatched 0s onto the stack, and q_3 to push unmatched 1's onto the stack.

PDA, Example 2 (cont'd)



Example: $(q_0, 011100, \epsilon) \Rightarrow (q_1, 011100, \$) \Rightarrow (q_2, 11100, \$) \Rightarrow (q_1, 11100, 0\$) \Rightarrow (q_1, 1100, \$) \Rightarrow (q_3, 100, \$) \Rightarrow (q_1, 100, 1\$) \Rightarrow (q_3, 00, 1\$) \Rightarrow (q_1, 00, 11\$) \Rightarrow (q_1, 0, 1\$) \Rightarrow (q_1, \epsilon, \$) \Rightarrow (q_4, \epsilon, \epsilon)$

PDAs Recognize CFLs

Theorem. *Each context-free language is recognized by a PDA.*

Proof Let L be produced by a Chomsky normal form grammar $G = (V, \Sigma, R, S)$.

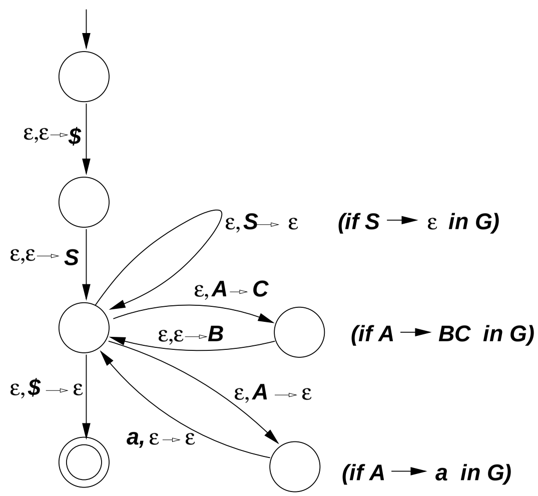
IDEA Construct a PDA that performs **leftmost derivations**. A step of a derivation using $A \to BC$ is implemented by **popping the left-hand side and pushing the right-hand side of the rule onto the stack**. A step using $A \to a$ is implemented by **popping the left-hand side and comparing the right-hand side to the input symbol (expending that symbol)**.

PDA Recognize CFLs (cont'd)

N's input alphabet is Σ and stack alphabet is V. It has one accept state.

N pushes a \$ and then an S on the stack and repeats the following until it halts:

1. N pops a stack symbol A. If $A = \$$, N **enters the accept state and halts**.
2. N **nondeterministically chooses a production rule $A \to w$.**
   - If $w = \epsilon$, N takes no action (back to step 1).
   - If $w = BC$ for some variables B, C, then N **pushes a C then a B onto the stack.**
   - If $w = a$, N reads the next input symbol b; **if $a \neq b$ then N halts.**



CFLs Capture PDA

Theorem. *Every language recognized by PDA is context free.*

Proof Let L be recognized by a PDA M. We can assume

(*) **M has a unique final state and, when it enters the state, the stack is empty.**

(**) **In a single move M either pops or pushes, but not both.**

Idea for ensuring (*): Modify M to create a new PDA N, which has

- a new stack symbol $\perp$ (to mark bottom of stack),
- a new initial state I (for putting $\perp$ on the stack),
- a clean-up state C (for clearing the stack),
- a new (unique) accepting state A.

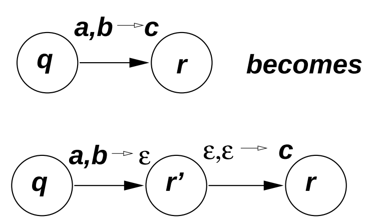
CFLs Capture PDA (cont'd)

- The only permissible action in I is to **put one $\perp$ on the stack without reading an input symbol and go to the old initial state.**
- In each state in Q the action of N is the same except that **there is an (ϵ, ϵ)-transition from each former accepting state to C.**
- The goal in state C is to **remove stack symbols one after another** and to **enter A upon observing a $\perp$.**

CFLs Capture PDA (cont'd)

As to (**), **divide each state involving push and pop into two states, one for push only and the other for pop only:**

$(r, c) \in \delta(q, a, b)$ becomes a pair of moves:
(r', ϵ) in $\delta(q, a, b)$ followed by (r, c) in $\delta(r', \epsilon, \epsilon)$



CFLs Capture PDA (cont'd)

Now suppose M satisfies (*) and (**).

Construct a CFG (V, Σ, P, S): $V = \{A_{pq} \mid p, q \in Q\}$ and $S = A_{q_0 f}$, where q_0 is the initial state of M and f is the unique final state of M.

Key idea: For any states p, q, A_{pq} is a variable generating **the strings that can take M from p to q while "preserving the stack"**. Then by taking p as the start state and q as the accept state, we have a variable that generates **precisely the strings accepted by M**, and thus can serve as start symbol.

CFLs Capture PDA (cont'd)

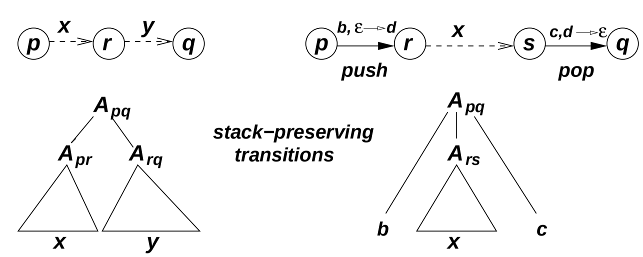
More precisely, the variable A_{pq} corresponds to the set of all strings expended by M under the following conditions:

- **M starts with p and ends with state q.**
- For some $k \geq 0$, the stack height is:
  - **always at least k** and
  - **precisely k at the start as well as at the end**.

Production rules:

- For every $p \in Q$, $A_{pp} \to \epsilon$.
- For every $p, q, r \in Q$, $A_{pq} \to A_{pr} A_{rq}$.
- For every $p, q, r, s \in Q$, $b, c \in \Sigma_\epsilon$, and $d \in \Gamma_\epsilon$,

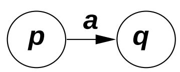
  if $(r, d) \in \delta(p, b, \epsilon)$ and $(q, \epsilon) \in \delta(s, c, d)$, then $A_{pq} \to b A_{rs} c$.



Regular Languages ⊂ CFLs

First proof Any FA can be viewed as a PDA that never pops or pushes the stack.

Second proof Build a CFG for a FA, where

for  ***we use*** ***P → aQ***

for  ***we use*** ***R → ε***

Properties of Context-Free Languages

Theorem. *The context-free languages are closed under union, concatenation, and star.*

Proof Let S_1 and S_2 be the start symbols of two CFGs. Let S be the new start symbol of the new CFG we are creating.

Adding $S \Rightarrow S_1 \mid S_2$ works for union.

Adding $S \Rightarrow S_1 S_2$ works for concatenation.

Adding $S \Rightarrow S S_1$ works for star. ∎

This also shows how regular expressions can be converted to equivalent CFGs, providing a 3rd proof that regular languages are CFLs.