Lecture 2-2: Regular Languages, continued

Closure Properties of Regular Languages

Theorem. *The regular languages are closed under complement, union, intersection, concatenation, and star.*

Proof The closure properties under union, concatenation, and star follow from the fact that **the regular languages are those that are expressible with regular expressions**.

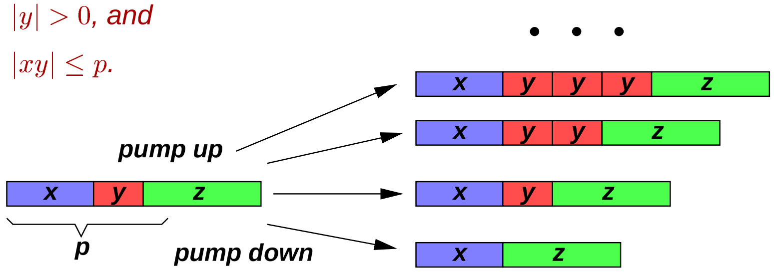
To prove the closure property under complement note that replacing the set of final states with its complement yields an FA for the complement.

Now the closure property under intersection follows by **de Morgan's Law.** ∎

The Pumping Lemma

Theorem. *(Pumping Lemma) Let L be a regular language. There exists a positive integer p such that for every $w \in L$ of length at least p, w is divided into three pieces, $w = xyz$, such that*

- *for each $i \geq 0$, $xy^iz \in L$,*
- *$|y| > 0$, and*
- *$|xy| \leq p$.*



Proof of the Pumping Lemma

Let L be recognized by an FA M. Let p be the number of states of M. Let $w, |w| = n \geq p$, be in L. Let $(q_0, q_1, \cdots, q_n)$ be the state sequence of M for accepting w.

> **The Pigeon Hole Principle:** *Suppose pigeons are placed in holes. If there are more pigeons than holes then some hole has to get at least two pigeons.*

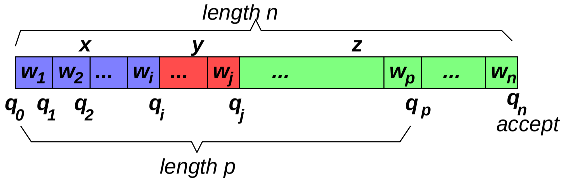
Since $n \geq p$, by the pigeon hole principle, for some i, j, $0 \leq i < j \leq n$, $q_i = q_j$. Pick smallest such pair (i, j). Then $j < p$.

Proof of the Pumping Lemma (cont'd)

Let $x = w_1 \cdots w_i$, $y = w_{i+1} \cdots w_j$, and $z = w_{j+1} \cdots w_n$. Then

- $|y| > 0$,
- $|xy| \leq p$,
- x takes M from q_0 to q_i,
- y takes M from q_i to q_i,
- z takes M from q_i to q_n, which is an accepting state.

So for every $i \geq 0$, xy^iz takes M from q_0 to q_n, so is a member of L. ▮



Application of the Pumping Lemma

Example 1: $B = \{0^n1^n \mid n \geq 0\}$ is not regular.

Proof By contradiction. Assume, on the contrary, that B is regular. Then the **pumping lemma** holds for B. Let p be a constant for which the lemma holds for B and $w = 0^p1^p$. Then w is in B and is divided into $w = xyz$ such that

- for each $i \geq 0$, $xy^iz \in B$,
- $|y| > 0$, and
- $|xy| \leq p$.

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Then x and y are all 0s. So, $xyyz$ has more 0s than 1s and is in B, a contradiction. So, B is not regular. ▮

Example 2: $C = \{w \mid w \in \{0,1\}^*$ and has an equal number of 0s and 1s $\}$ is not regular.

Proof Assume, on the contrary, that C is regular.

Let p be a constant for which the pumping lemma holds for C.

Let $w = 0^p1^p$. Then $w = xyz$ such that $|xy| \leq p$, $|y| \geq 1$, and for every $i \geq 0$, $xy^iz \in C$. Here $y \in 0^*$ since w begins with 0^p.

Pick $i = 2$, then $0^{p+|y|}1^p \in C$, a contradiction.

Alternatively, let $C' = C \bigcap 0^*1^*$. If C were regular, then C' would be regular. But $C' = B$ in Example 1. So, C' is not regular. ▮

Example 3: $F = \{ww \mid w \in \{0,1\}^*\}$ is not regular.

Proof Assume, on the contrary, that F is regular.

Let p be a constant for which the pumping lemma holds for F.

Let $w = 0^p1^p0^p1^p$. Then, w is divided into $w = xyz$ such that $|y| > 0$, $|xy| \leq p$, and $(\forall i \geq 0)[xy^iz \in F]$. Here $y \in 0^*$ since w begins with 0^p.

Pick $i = 0$, we have $0^q1^p0^p1^p \in F$, where $q < p$, a contradiction. ▮

Example 4: $D = \{1^{n^2} \mid n \geq 0\}$ is not regular.

Proof Assume, on the contrary, that D is regular.

Let p be a constant for which the pumping lemma holds for D.

Let $w = 1^{p^2}$. Then $w = xyz$ for some x, y, z such that $|y| > 0$, $|xy| \leq p$, and $(\forall i \geq 0)[xy^iz \in D]$.

Let $l = |y|$. Then $0 < l < p$. By plugging in $i = 2$, we have $1^{p^2+l} \in D$, but $p^2 + l < (p+1)^2$, a contradiction. ∎

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Example 5: $E = \{0^i1^j \mid i > j\}$ is not regular.

Proof Assume, on the contrary, that E is regular.

Let p be a constant for which the pumping lemma holds for E.

Let $w = 0^p1^{p-1}$. Then $w = xyz$ for some x, y, z such that $|y| > 0$, $|xy| \leq p$, and $(\forall i \geq 0)[xy^iz \in D]$. Here $y \in 0^*$ since the first p symbols of w are all 0.

With $i = 0$, we have $0^q1^{p-1} \in E$, where $q \leq p - 1$, a contradiction. ∎

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