# Lecture 2-1: Regular Languages



## Finite Automata

A **finite automaton** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

1. $Q$ is a finite set called the **states**,
2. $\Sigma$ is a finite set called the **alphabet**,
3. $\delta : Q \times \Sigma \longrightarrow Q$ is the **transition function**,
4. $q_0 \in Q$ is the **initial state**, and
5. $F \subseteq Q$ is the **set of accepting states**.

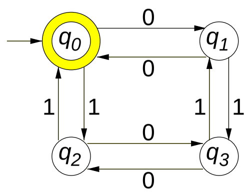
Let $M = (Q, \Sigma, \delta, q_0, F)$ be an FA. A string $w = w_1 \cdots w_n$ is **accepted** by $M$ if there exists a sequence $(p_0, \ldots, p_n)$ of states in $Q$ such that $p_0 = q_0$, $p_n \in F$, and for every $i$, $1 \leq i \leq n$, $\delta(p_{i-1}, w_i) = p_i$.



The language **recognized** (or **accepted**) by $M$, notated as $L(M)$, is the language over $\Sigma$ such that

**(*)** for every string $w$ over $\Sigma$, $w \in L(M) \Leftrightarrow M$ accepts $w$.

**Example:** A FA that recognizes the language over $\{0, 1\}$ consisting of all the strings with an even number of $0$s and an even number of $1$s






## Regular Languages

Let $A$ and $B$ be two languages. Define:

- **Union** of $A$ and $B$, $A \cup B$, is $\{x \mid x \in A \text{ or } x \in B\}$,
- **Concatenation** of $A$ and $B$, $A \circ B$, is $\{xy \mid x \in A \text{ and } y \in B\}$,
- **Star** of $A$, $A^*$, is $\{x_1 x_2 \cdots x_k \mid k \geq 0 \text{ and } x_1, \ldots, x_k \in A\}$.

The class of **regular languages** is the class of languages recognized by finite automata.

## Nondeterministic Finite Automata

A **nondeterministic finite automaton** is a 5-tuple $N = (Q, \Sigma, \delta, q_0, F)$, where $\delta$ now is a mapping of $Q \times \Sigma_\epsilon$ to $\mathcal{P}(Q)$, the **power set of $Q$**, i.e., the collection of all subsets of $Q$, where $\Sigma_\epsilon = \Sigma \cup \{\epsilon\}$.

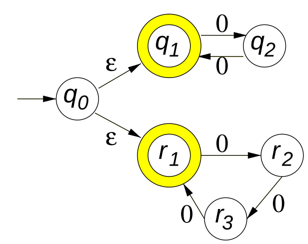
For each $p \in Q$ and each $a \in \Sigma_\epsilon$, "$\delta(s, a) = R$" states that **upon reading an $a$, $N$ can go from $s$ to any state in $R$.**

A string $w = w_1 \cdots w_n$ is **accepted** by $N$ if there exists a sequence $(p_0, \ldots, p_m)$ of states in $Q$ and a representation $y = y_1 \cdots y_m$ of $w$ over $\Sigma_\epsilon$ such that $p_0 = q_0$ and for every $i$, $1 \le i \le m$, $p_i \in \delta(p_{i-1}, y_i)$.



## Example of NFA

An NFA that recognizes the language over $\{0\}$ that consists of all strings $w$ such that $|w|$ is either a multiple of 2 or a multiple of 3.






## FA = NFA

**Theorem.** *Every NFA can be converted an equivalent FA.*

**Proof** Let $N = (Q, \Sigma, \delta, q_0, F)$ be an NFA. Define a FA $M = (S, \Sigma, \gamma, s_0, G)$ by

- $S = \mathcal{P}(Q)$,
- $s_0 = \{q_0\}$,
- $G = \{A \subseteq Q \mid A \cap F \neq \emptyset\}$, and
- for each $A \in S$ and $b \in \Sigma$, $\gamma(A, b) = \bigcup_{p \in A} \delta(p, \epsilon^* b \epsilon^*)$.

Here $\gamma(A, b)$ is the set of all states that $M$ can go to from one of the states in $A$, upon receiving symbol $b$. So, $w$ over $\Sigma$ is accepted by $N$ if and only if $w$ takes $M$ from the state $s_0$ to a subset of $Q$ containing an element in $F$ (the set of such subsets is $G$). ∎



## Regular Expression

An expression $R$ is a **regular expression** if $R$ is

1. $a$ for some $a$ in some alphabet $\Sigma$,
2. $\epsilon$,
3. $\emptyset$,
4. $(R_1 \cup R_2)$ for some regular expressions $R_1$ and $R_2$,
5. $(R_1 \circ R_2)$ for some regular expressions $R_1$ and $R_2$, or
6. $(R_1)^*$ for some regular expression $R_1$.

## Finite Automata are equivalent to Regular Expressions

This requires proofs in both directions.

**Lemma (REXPR ⇒ NFA ⇒ FA).** *Every regular expression describes a regular language.*

**Proof** Each set consisting only of a single letter is regular, the set consisting only of the empty string is regular, and the empty set is regular.

Let $L_i = L(M_i), i = 1, 2$, be a FA with initial state $p_i$ and final states $F_i$. Construct an NFA with initial state $p_0$ and final states $F_0$:

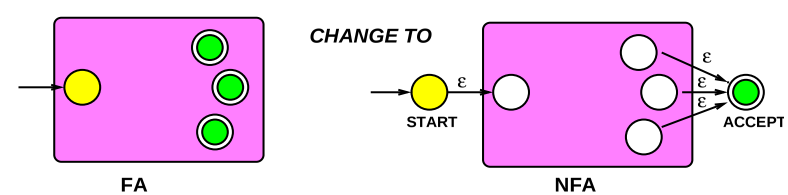
**Union:** $p_0$ has an $\epsilon$-move to $p_1$ and to $p_2$; $F_0 = F_1 \cup F_2$
**Concatenation of $L_1$ and $L_2$:** $p_0 = q_1$, an $\epsilon$-move from each $s \in F_1$ to $p_2$, $F_0 = F_2$
**Star of $L_1$:** $p_0 = p_1$, an $\epsilon$-move from each $s \in F_1$ to $p_1$, $F_0 = F_1$



**Lemma (FA ⇒ NFA ⇒ REXPR).** *Every regular language is described by some regular expression.*

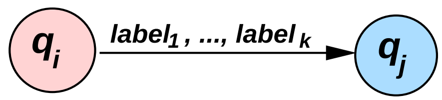
**Proof** (Sketch) We want to get from an FA to a regular expression.

**Step 1:** Change an FA for any given regular language $L$ to an equivalent NFA with a new start state and a new, unique accepting state, using $\epsilon$-transitions.

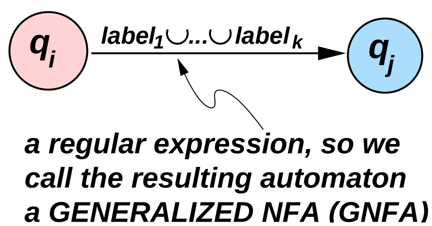





**Step 2:** **Combine multiple labels.**
(a) For every transition

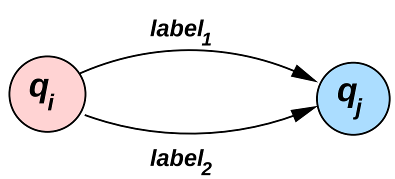



in the NFA, change it to

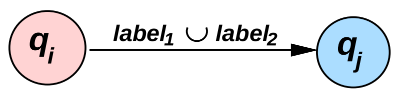




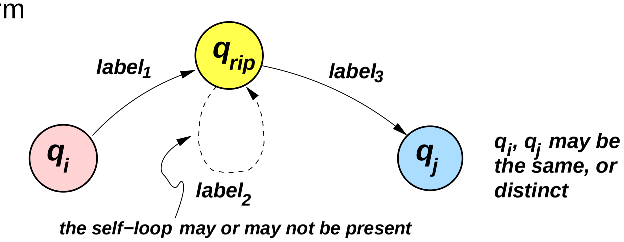

(b) For every pair of transitions (while there remain such pairs)



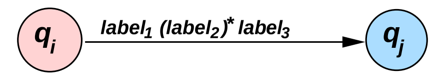

in the GNFA, change these to



**Step 2:** **Eliminate ("rip out") successive states of the GNFA (other than START and ACCEPT)**, replacing lost 2-step paths by 1-step paths. I.e., when we eliminate a state $q_{rip}$, we generally "lose" some local paths of the form

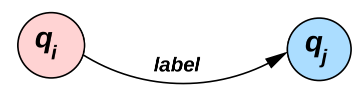



For *each* such local path,
(a) insert a single arrow labeled like this:

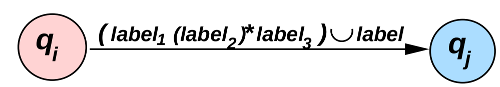





(b) If there was already a transition

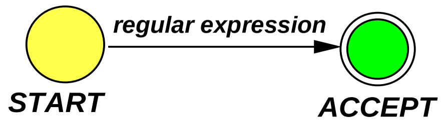



from $q_i$ to $q_j$, merge the two arrows into






**Result:**

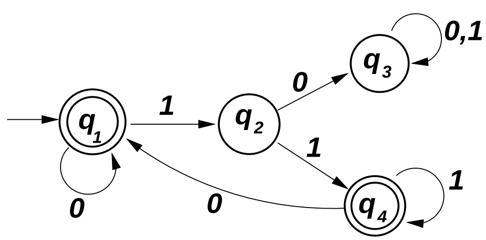



The regular expression describes $L$, the language recognized by the original FA. Showing this requires an inductive proof that *at each step* of the conversion, the resulting GNFA still accepts exactly the same input sequences as before. We'll omit further details.

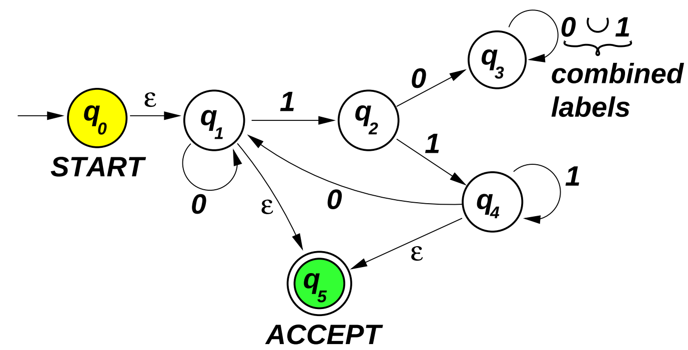


## Example of FA $\Rightarrow$ REXPR

A FA that recognizes the language over $\{0, 1\}$ that consists of all strings $w$ with no "isolated" 1s, i.e., wherever there is a 1, there is another 1 adjacent on its left or right.

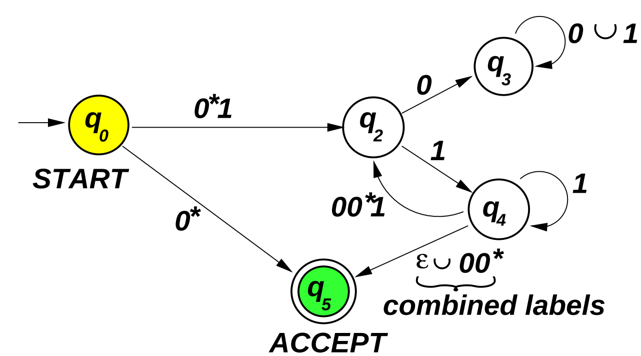


**Steps 1 and 2:**



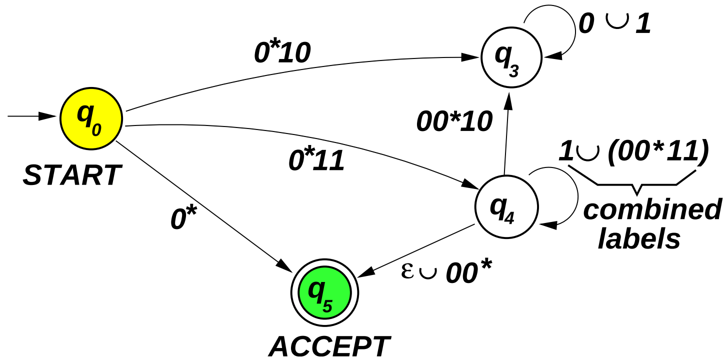



**Step 3:**
Rip $q_1$: there are 4 local paths to replace





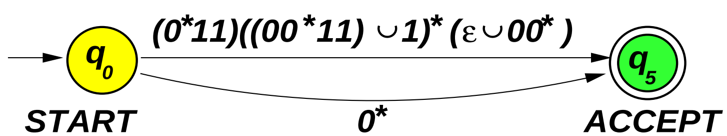

**Step 3, continued:**
Rip $q_2$: there are 4 local paths to replace



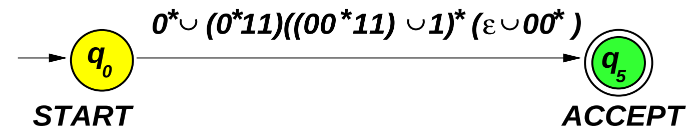



**Step 3, continued:**
Rip $q_3$: there are no local paths through $q_3$, so it just disappears

Rip $q_4$: there is 1 local path to replace



**Step 3, concluded:**
Combine labels:




Not as simple as it might be: $(111^* \cup 0)^* \cup 1$