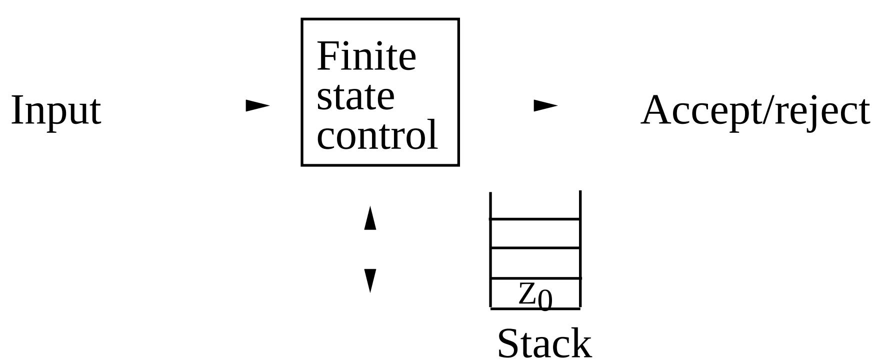
Pushdown Automata

A pushdown automaton (PDA) is essentially an ϵ-NFA with a stack.

On a transition the PDA:

1. Consumes an input symbol. or ε
2. Goes to a new state (or stays in the old).
3. Replaces the top of the stack by any string (does nothing, pops the stack, or pushes a string onto the stack)



Example: Let's consider

$$L_{wwr} = \{ww^R : w \in \{0,1\}^*\},$$

with "grammar" $P \to 0P0$, $P \to 1P1$, $P \to \epsilon$. A PDA for L_{wwr} has three states, and operates as follows:

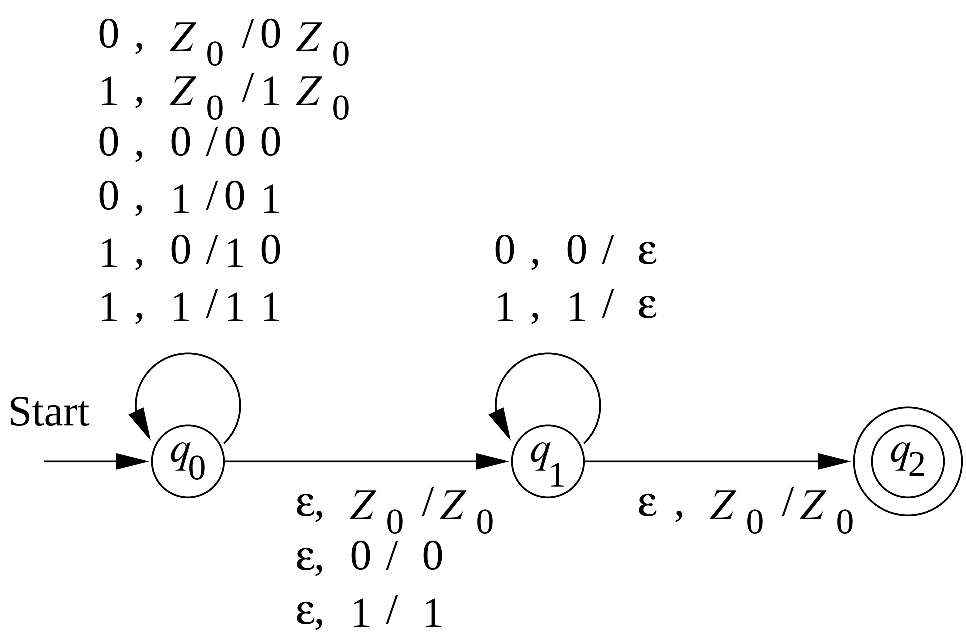
1. Guess that you are reading w. Stay in state 0, and push the input symbol onto the stack.

2. Guess that you're in the middle of ww^R. Go spontanteously to state 1.

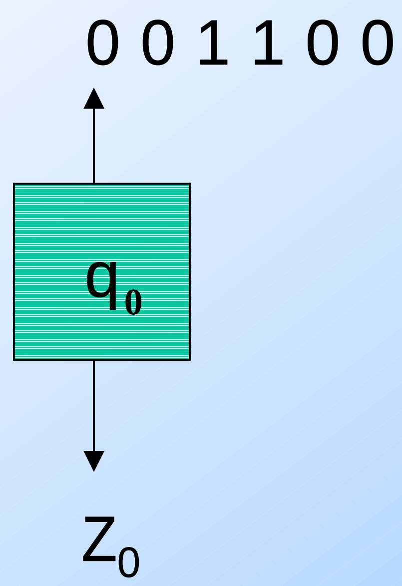
3. You're now reading the head of w^R. Compare it to the top of the stack. If they match, pop the stack, and remain in state 1. If they don't match, go to sleep.

4. If the stack is empty, go to state 2 and accept.

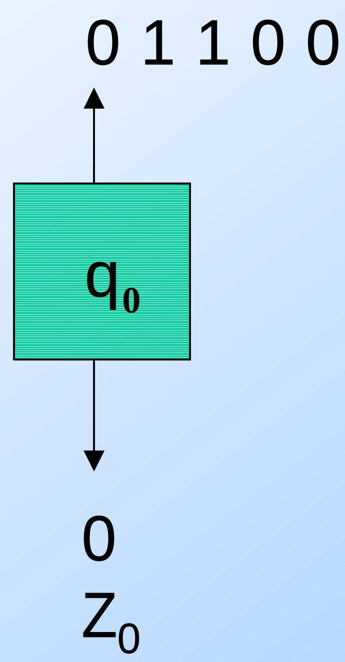
The PDA for L_{wwr} as a transition diagram:



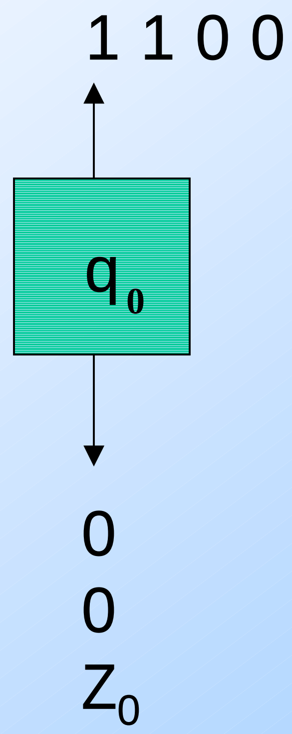
Actions of the Example PDA



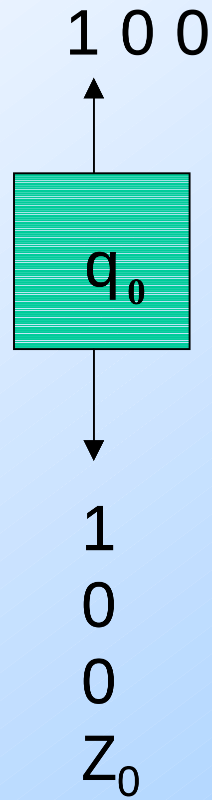
Actions of the Example PDA



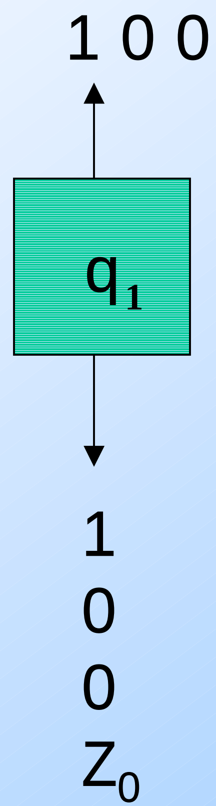
Actions of the Example PDA



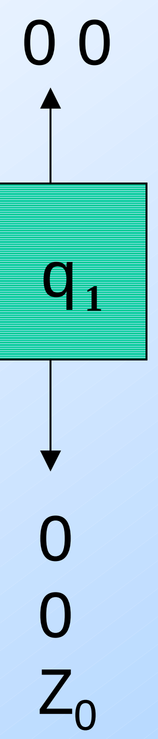
Actions of the Example PDA



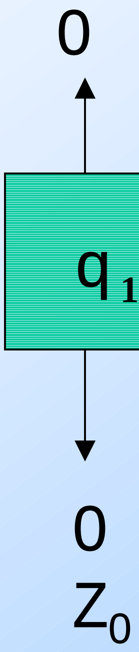
Actions of the Example PDA



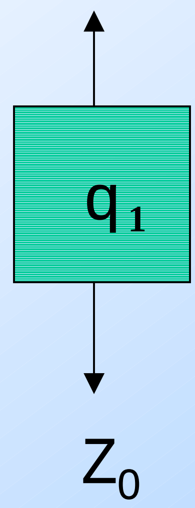
Actions of the Example PDA



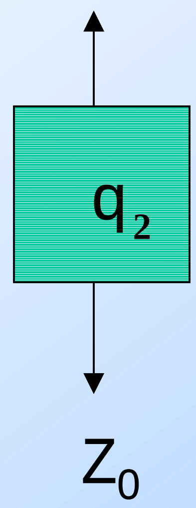
Actions of the Example PDA



Actions of the Example PDA



Actions of the Example PDA



PDA formally

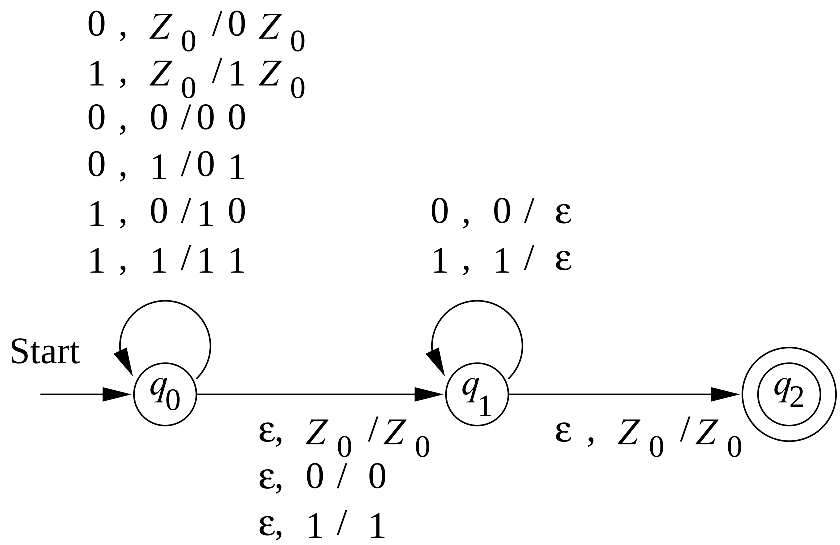
A PDA is a seven-tuple:

$$P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F),$$

where

- Q is a finite set of states,
- Σ is a finite *input alphabet,*
- Γ is a finite *stack alphabet,*
- $\delta : Q \times (\Sigma \cup \{\epsilon\}) \times \Gamma \to 2^{Q \times \Gamma^*}$ is the *transition function,*
- q_0 is the *start state,*
- $Z_0 \in \Gamma$ is the *start symbol* for the stack, and
- $F \subseteq Q$ is the set of *accepting states.*

Example: The PDA



is actually the seven-tuple

$$P = (\{q_0, q_1, q_2\}, \{0, 1\}, \{0, 1, Z_0\}, \delta, q_0, Z_0, \{q_2\}),$$

where δ is given by the following table (set brackets missing):

	$0, Z_0$	$1, Z_0$	0,0	0,1	1,0	1,1	ϵ, Z_0	$\epsilon, 0$	$\epsilon, 1$
$\rightarrow q_0$	$q_0, 0Z_0$	$q_0, 1Z_0$	$q_0, 00$	$q_0, 01$	$q_0, 10$	$q_0, 11$	q_1, Z_0	$q_1, 0$	$q_1, 1$
q_1			q_1, ϵ			q_1, ϵ	q_2, Z_0		
$\star q_2$									

Instantaneous Descriptions

A PDA goes from configuration to configuration when consuming input.

To reason about PDA computation, we use *instantaneous descriptions* of the PDA. An ID is a triple

$$(q, w, \gamma)$$

where q is the state, w the remaining input, and γ the stack contents.

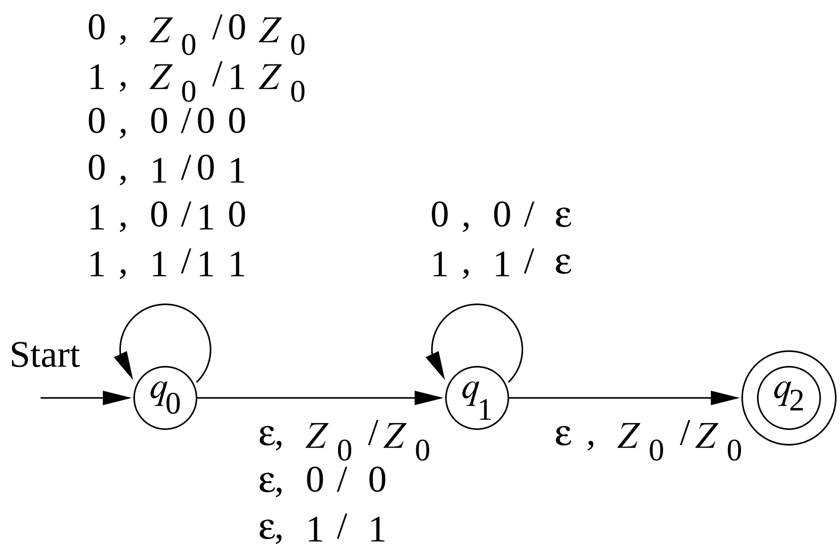
Let $P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ be a PDA. Then $\forall w \in \Sigma^*, \beta \in \Gamma^*$:

$$(p, \alpha) \in \delta(q, a, X) \Rightarrow (q, aw, X\beta) \vdash (p, w, \alpha\beta).$$

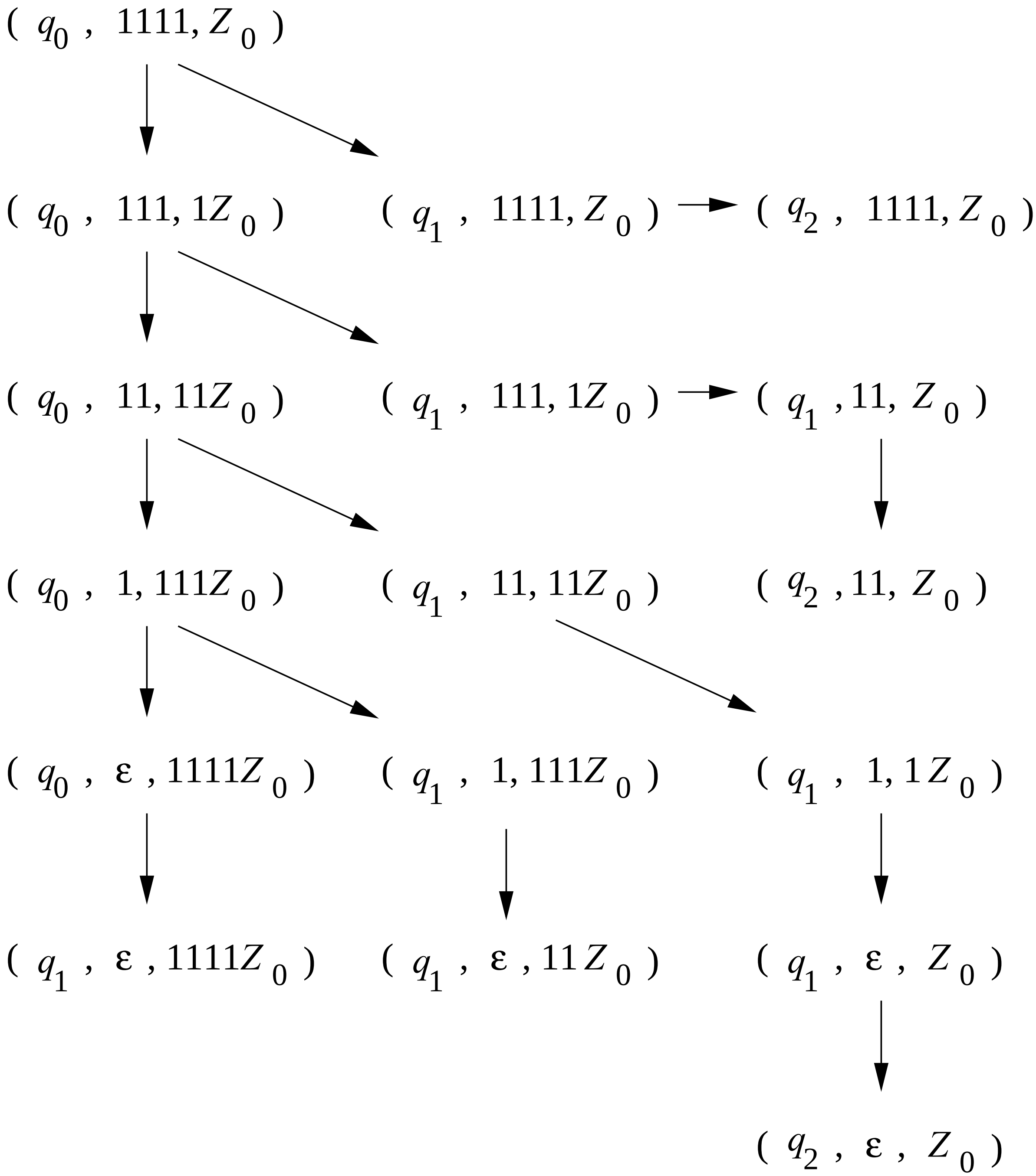
yield

We define $\overset{*}{\vdash}$ to be the reflexive-transitive closure of $\vdash$.

Example: On input 1111 the PDA



has the following computation sequences:


(q0 , 1111, Z0)
(q0 , 111, 1Z0)
(q1 , 1111, Z0)
(q2 , 1111, Z0)
(q0 , 11, 11Z0)
(q1 , 111, 1Z0)
(q1 ,11, Z0)
(q0 , 1, 111Z0)
(q1 , 11, 11Z0)
(q2 ,11, Z0)
(q0 , ε , 1111Z0)
(q1 , 1, 111Z0)
(q1 , 1, 1 Z0)
(q1 , ε , 1111Z0)
(q1 , ε , 11 Z0)
(q1 , ε , Z0)
(q2 , ε , Z0)

The following properties hold:

1. If an ID sequence is a legal computation for a PDA, then so is the sequence obtained by adding an additional string at the end of component number two.

2. If an ID sequence is a legal computation for a PDA, then so is the sequence obtained by adding an additional string at the bottom of component number three.

3. If an ID sequence is a legal computation for a PDA, and some tail of the input is not consumed, then removing this tail from all ID's result in a legal computation sequence.

Theorem 6.5: $\forall w \in \Sigma^*,\ \gamma \in \Gamma^*$:

$$(q, x, \alpha) \overset{*}{\vdash} (p, y, \beta) \Rightarrow (q, xw, \alpha\gamma) \overset{*}{\vdash} (p, yw, \beta\gamma).$$

Proof: Induction on the length of the sequence to the left.

Note: If $\gamma = \epsilon$ we have proerty 1, and if $w = \epsilon$ we have property 2.

Note2: The reverse of the theorem is false.

For property 3 we have

Theorem 6.6:

$$(q, xw, \alpha) \overset{*}{\vdash} (p, yw, \beta) \Rightarrow (q, x, \alpha) \overset{*}{\vdash} (p, y, \beta).$$

Acceptance by final state

Let $P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ be a PDA. The *language accepted by P by final state* is

$$L(P) = \{w : (q_0, w, Z_0) \overset{*}{\vdash} (q, \epsilon, \alpha),\ q \in F\}.$$

Example: The PDA on slide 183 accepts exactly L_{wwr}.

Let P be the machine. We prove that $L(P) = L_{wwr}$.

($\supseteq$-*direction*.) Let $x \in L_{wwr}$. Then $x = ww^R$, and the following is a legal computation sequence

$$(q_0, ww^R, Z_0) \overset{*}{\vdash} (q_0, w^R, w^R Z_0) \vdash (q_1, w^R, w^R Z_0) \overset{*}{\vdash}$$

$$(q_1, \epsilon, Z_0) \vdash (q_2, \epsilon, Z_0).$$

($\subseteq$-*direction.*)

Observe that the only way the PDA can enter q_2 is if it is in state q_1 with **top stack symbol = z_0**

Thus it is sufficient to show that if $(q_0, x, Z_0) \overset{*}{\vdash} (q_1, \epsilon, Z_0)$ then $x = ww^R$, for some word w.

We'll show by induction on $|x|$ that

$$(q_0, x, \alpha) \overset{*}{\vdash} (q_1, \epsilon, \alpha) \;\; \Rightarrow \;\; x = ww^R.$$

Basis: If $x = \epsilon$ then x is a palindrome.

Induction: Suppose $x = a_1 a_2 \cdots a_n$, where $n > 0$, and the IH holds for shorter strings.

Ther are two moves for the PDA from ID (q_0, x, α):

Move 1: The spontaneous $(q_0, x, \alpha) \vdash (q_1, x, \alpha)$. Now $(q_1, x, \alpha) \overset{*}{\vdash} (q_1, \epsilon, \beta)$ implies that $|\beta| < |\alpha|$, which implies $\beta \neq \alpha$.

Move 2: Loop and push $(q_0, a_1 a_2 \cdots a_n, \alpha) \vdash (q_0, a_2 \cdots a_n, a_1\alpha)$.

In this case there is a sequence

$(q_0, a_1 a_2 \cdots a_n, \alpha) \vdash (q_0, a_2 \cdots a_n, a_1\alpha) \vdash \cdots \vdash (q_1, a_n, a_1\alpha) \vdash (q_1, \epsilon, \alpha)$.

Thus $a_1 = a_n$ and

$$(q_0, a_2 \cdots a_n, a_1\alpha) \overset{*}{\vdash} (q_1, a_n, a_1\alpha).$$

By Theorem 6.6 we can remove a_n. Therefore

$$(q_0, a_2 \cdots a_{n-1}, a_1\alpha \overset{*}{\vdash} (q_1, \epsilon, a_1\alpha).$$

Then, by the IH $a_2 \cdots a_{n-1} = yy^R$. Then $x = a_1 yy^R a_n$ is a palindrome.

Give a final-state PDA for balanced brackets (or Dyck language): B -> BB | (B) | ε

$L_2 = \{0^m 1^n 2^p \mid m,n, p >= 0, m+n = p\}$

Acceptance by Empty Stack

Let $P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ be a PDA. The *language accepted by P by empty stack* is

$$N(P) = \{w : (q_0, w, Z_0) \overset{*}{\vdash} (q, \epsilon, \epsilon)\}.$$

Note: q can be any state.

Question: How to modify the palindrome-PDA to accept by empty stack? two ways to do it!

Give an empty-stack PDA for balanced brackets (or Dyck language): B -> BB | (B) | ε

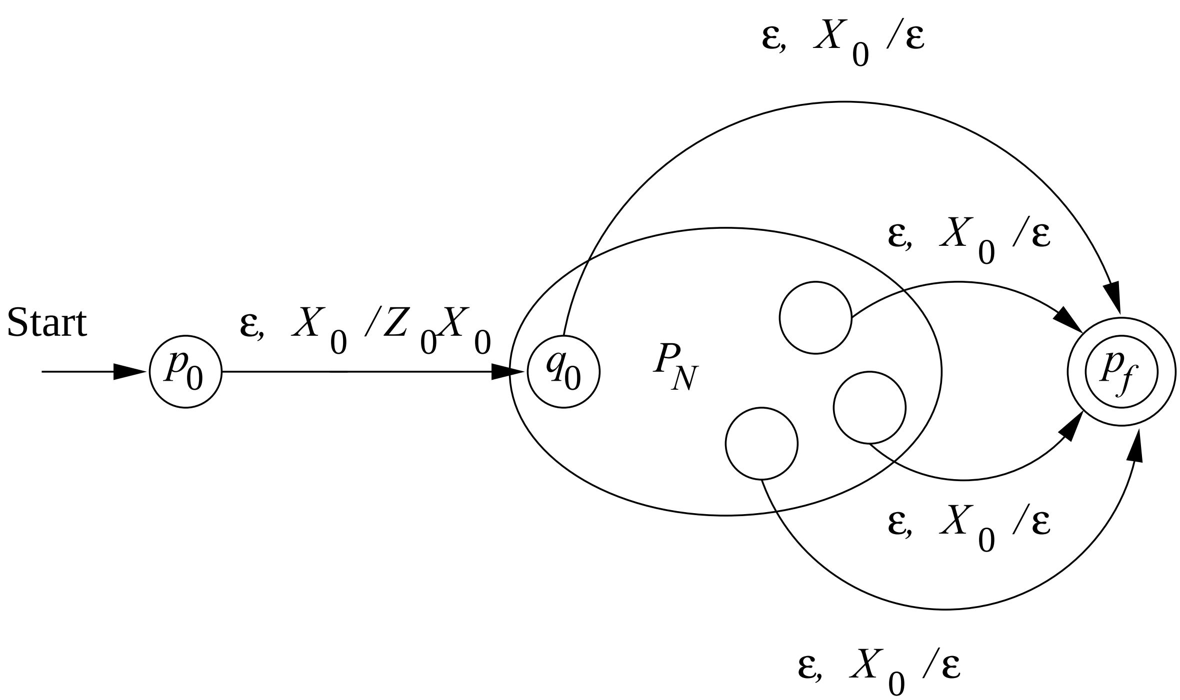
From Empty Stack to Final State

Theorem 6.9: If $L = N(P_N)$ for some PDA $P_N = (Q, \Sigma, \Gamma, \delta_N, q_0, Z_0)$, then $\exists$ PDA P_F, such that $L = L(P_F)$.

Proof: Let

$$P_F = (Q \cup \{p_0, p_f\}, \Sigma, \Gamma \cup \{X_0\}, \delta_F, p_0, X_0, \{p_f\})$$

where $\delta_F(p_0, \epsilon, X_0) = \{(q_0, Z_0X_0)\}$, and for all $q \in Q, a \in \Sigma \cup \{\epsilon\}, Y \in \Gamma : \delta_F(q, a, Y) = \delta_N(q, a, Y)$, and in addition $(p_f, \epsilon) \in \delta_F(q, \epsilon, X_0)$.



We have to show that $L(P_F) = N(P_N)$.

($\supseteq$*direction.*) Let $w \in N(P_N)$. Then

$$(q_0, w, Z_0) \overset{*}{\underset{N}{\vdash}} (q, \epsilon, \epsilon),$$

for some q. From Theorem 6.5 we get

$$(q_0, w, Z_0 X_0) \overset{*}{\underset{N}{\vdash}} (q, \epsilon, X_0).$$

Since $\delta_N \subset \delta_F$ we have

$$(q_0, w, Z_0 X_0) \overset{*}{\underset{F}{\vdash}} (q, \epsilon, X_0).$$

We conclude that

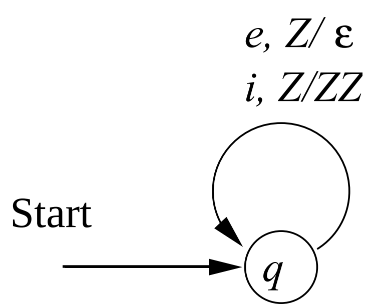
$$(p_0, w, X_0) \underset{F}{\vdash} (q_0, w, Z_0 X_0) \overset{*}{\underset{F}{\vdash}} (q, \epsilon, X_0) \underset{F}{\vdash} (p_f, \epsilon, \epsilon).$$

($\subseteq$*direction.*) By inspecting the diagram.

Let's design P_N for for catching errors in strings meant to be in the *if-else*-grammar G

$$S \rightarrow \epsilon | SS | iS | iSe.$$

Here e.g. $\{ieie, iie, iei\} \subseteq$ **L(G)** and e.g. $\{ei, ieeii\} \cap$ **L(G)** $= \emptyset$. The diagram for P_N is



Note that this PDA does not really accept the complement of L(G); it gets "stuck" as soon it detects the first excess "e".

Formally,

$$P_N = (\{q\}, \{i, e\}, \{Z\}, \delta_N, q, Z),$$

where $\delta_N(q, i, Z) = \{(q, ZZ)\}$,
and $\delta_N(q, e, Z) = \{(q, \epsilon)\}$.

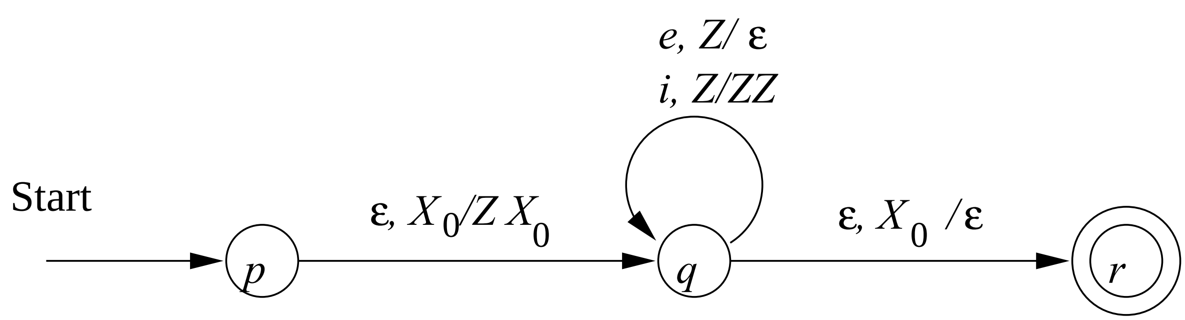
Question: Does one state suffice for empy-stack PDAs?

From P_N we can construct

$$P_F = (\{p,q,r\},\{i,e\},\{Z,X_0\},\delta_F,p,X_0,\{r\}),$$

where
$\delta_F(p,\epsilon,X_0) = \{(q,ZX_0)\}$,
$\delta_F(q,i,Z) = \delta_N(q,i,Z) = \{(q,ZZ)\}$,
$\delta_F(q,e,Z) = \delta_N(q,e,Z) = \{(q,\epsilon)\}$, and
$\delta_F(q,\epsilon,X_0) = \{(r,\epsilon)\}$

The diagram for P_F is



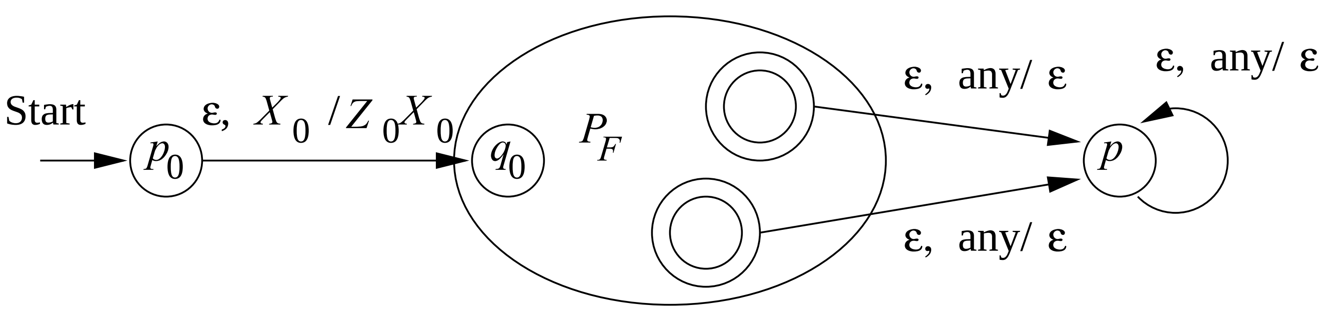
From Final State to Empty Stack

Theorem 6.11: Let $L = L(P_F)$, for some PDA $P_F = (Q, \Sigma, \Gamma, \delta_F, q_0, Z_0, F)$. Then $\exists$ PDA P_n, such that $L = N(P_N)$.

Proof: Let

$$P_N = (Q \cup \{p_0, p\}, \Sigma, \Gamma \cup \{X_0\}, \delta_N, p_0, X_0)$$

where $\delta_N(p_0, \epsilon, X_0) = \{(q_0, Z_0X_0)\}$, $\delta_N(p, \epsilon, Y) = \{(p, \epsilon)\}$, for $Y \in \Gamma \cup \{X_0\}$, and for all $q \in Q$, $a \in \Sigma \cup \{\epsilon\}, Y \in \Gamma$: $\delta_N(q, a, Y) = \delta_F(q, a, Y)$, and in addition $\forall q \in F$, and $Y \in \Gamma \cup \{X_0\}$: $(p, \epsilon) \in \delta_N(q, \epsilon, Y)$.



We have to show that $N(P_N) = L(P_F)$.

($\subseteq$-*direction*.) By inspecting the diagram.

($\supseteq$-*direction*.) Let $w \in L(P_F)$. Then

$$(q_0, w, Z_0) \overset{*}{\underset{F}{\vdash}} (q, \epsilon, \alpha),$$

for some $q \in F, \alpha \in \Gamma^*$. Since $\delta_F \subseteq \delta_N$, and Theorem 6.5 says that X_0 can be slid under the stack, we get

$$(q_0, w, Z_0X_0) \overset{*}{\underset{N}{\vdash}} (q, \epsilon, \alpha X_0).$$

The P_N can compute:

$$(p_0, w, X_0) \underset{N}{\vdash} (q_0, w, Z_0X_0) \overset{*}{\underset{N}{\vdash}} (q, \epsilon, \alpha X_0) \overset{*}{\underset{N}{\vdash}} (p, \epsilon, \epsilon).$$

Ex. Construct an empty-stack PDA for $L_3 = \{w \mid w \; \varepsilon \; \{0,1\}^*, w <> w^R\}$.

Equivalence of PDA's and CFG's

A language is

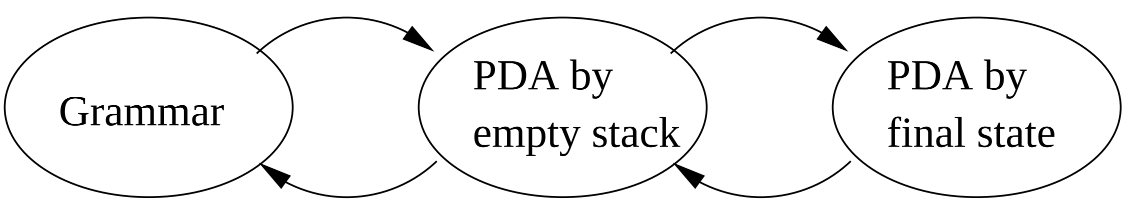
generated by a CFG

if and only if it is

accepted by a PDA by empty stack

if and only if it is

accepted by a PDA by final state



We already know how to go between null stack and final state.

From CFG's to PDA's

Given G, we construct a PDA that simulates $\overset{*}{\underset{lm}{\Rightarrow}}$.

We write left-sentential forms as

$$xA\alpha$$

where A is the leftmost variable in the form. For instance,

$$\underbrace{(a+}_{x}\underbrace{\underbrace{E}_{A}\underbrace{)}_{\alpha}}_{\text{tail}}$$

Let $xA\alpha \underset{lm}{\Rightarrow} x\beta\alpha$. This corresponds to the PDA first having consumed x and having $A\alpha$ on the stack, and then on ϵ it pops A and pushes β.

More fomally, let y, s.t. $w = xy$. Then the PDA goes non-deterministically from configuration $(q, y, A\alpha)$ to configuration $(q, y, \beta\alpha)$.

At $(q, y, \beta\alpha)$ the PDA behaves as before, unless there are terminals in the prefix of β. In that case, the PDA pops them, provided it can consume matching input.

If all guesses are right, the PDA ends up with empty stack and input.

Formally, let $G = (V, T, Q, S)$ be a CFG. Define P_G as

$$(\{q\}, T, V \cup T, \delta, q, S),$$

where

$$\delta(q, \epsilon, A) = \{(q, \beta) : A \to \beta \ \in \ Q\},$$

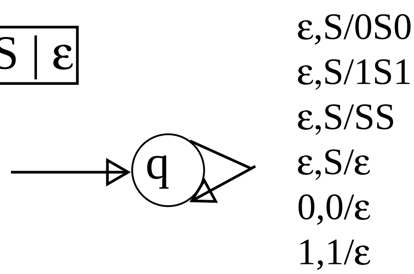
for $A \in V$, and

$$\delta(q, a, a) = \{(q, \epsilon)\},$$

for $a \in T$.

Example: On blackboard in class.

S -> 0S0 | 1S1 | SS | ε



Theorem 6.13: $N(P_G) = L(G)$.

Proof:

($\supseteq$-*direction*.) Let $w \in L(G)$. Then

$$S = \gamma_1 \underset{lm}{\Rightarrow} \gamma_2 \underset{lm}{\Rightarrow} \cdots \underset{lm}{\Rightarrow} \gamma_n = w$$

Let $\gamma_i = x_i \alpha_i$. We show by induction on i that

where x_i is a string of terminals and α_i begins with a variable

$$(q, w, S) \overset{*}{\vdash} (q, y_i, \alpha_i),$$

where $w = x_i y_i$.

Basis: For $i = 1, \gamma_1 = S$. Thus $x_1 = \epsilon$, and $y_1 = w$. Clearly $(q, w, S) \overset{*}{\vdash} (q, w, S)$.

Induction: IH is $(q, w, S) \overset{*}{\vdash} (q, y_i, \alpha_i)$. We have to show that

$$(q, y_i, \alpha_i) \overset{*}{\vdash} (q, y_{i+1}, \alpha_{i+1})$$

Now α_i begins with a variable A, and we have the form

$$\underbrace{x_i A \chi}_{\gamma_i} \underset{lm}{\Rightarrow} \underbrace{x_i \;\; \beta \chi \;\;}_{\gamma_{i+1}}$$

By IH $A\chi$ is on the stack, and y_i is unconsumed. From the construction of P_G it follows that we can make the move

$$(q, y_i, A\chi) \vdash (q, y_i, \beta\chi).$$

because x_{i+1} is a prefix of w

If β has a prefix of terminals, we can pop them with matching terminals in a prefix of y_i, ending up in configuration $(q, y_{i+1}, \alpha_{i+1})$, where α_{i+1} is the tail of the sentential form

$$x_{i+1}\alpha_{i+1} = \gamma_{i+1}.$$

Finally, since $\gamma_n = w$, we have $\alpha_n = \epsilon$, and $y_n = \epsilon$, and thus $(q, w, S) \overset{*}{\vdash} (q, \epsilon, \epsilon)$, i.e. $w \in N(P_G)$

($\subseteq$-*direction*.) We shall show by an induction on the length of $\overset{*}{\vdash}$, that

(♣) If $(q, x, A) \overset{*}{\vdash} (q, \epsilon, \epsilon)$, then $A \overset{*}{\Rightarrow} x$.

Basis: Length 1. Then it must be that $A \rightarrow \epsilon$ is in G, and we have $(q, \epsilon) \in \delta(q, \epsilon, A)$. Thus $A \overset{*}{\Rightarrow} \epsilon$.

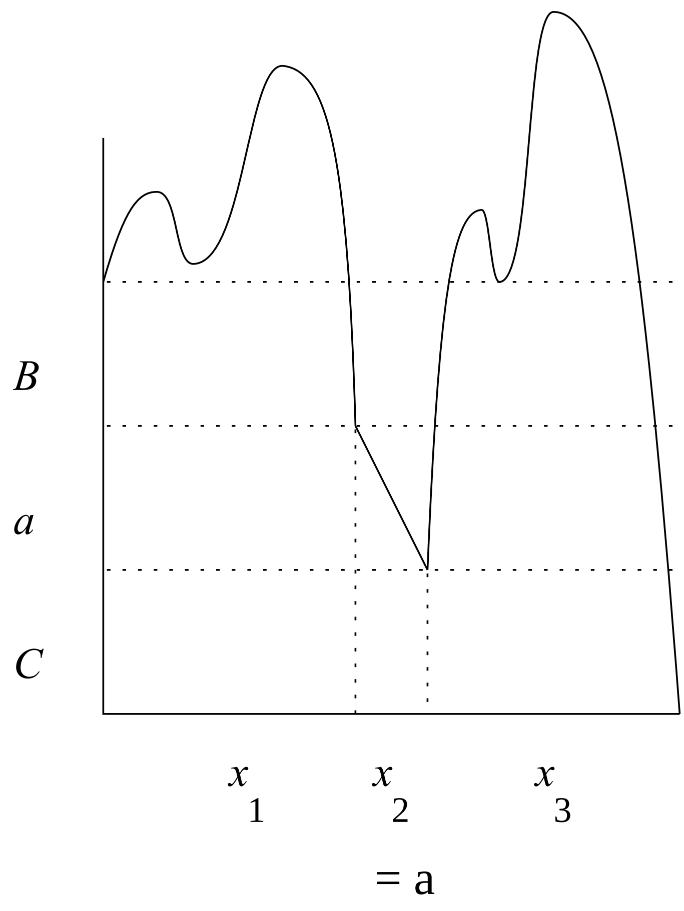
Induction: Length is $n > 1$, and the IH holds for lengths $< n$.

Since A is a variable, we must have

$$(q, x, A) \vdash (q, x, Y_1 Y_2 \cdots Y_k) \vdash \cdots \vdash (q, \epsilon, \epsilon)$$

where $A \rightarrow Y_1 Y_2 \cdots Y_k$ is in G.

We can now write x as $x_1 x_2 \cdots x_k$, according to the figure below, where $Y_1 = B, Y_2 = a$, and $Y_3 = C$.



Now we can conclude that

$$(q, x_i x_{i+1} \cdots x_k, Y_i) \overset{*}{\vdash} (q, x_{i+1} \cdots x_k, \epsilon)$$

in less than n steps, for all $i \in \{1, \ldots, k\}$. If Y_i is a variable we have by the IH and Theorem 6.6 that

$$Y_i \overset{*}{\Rightarrow} x_i$$

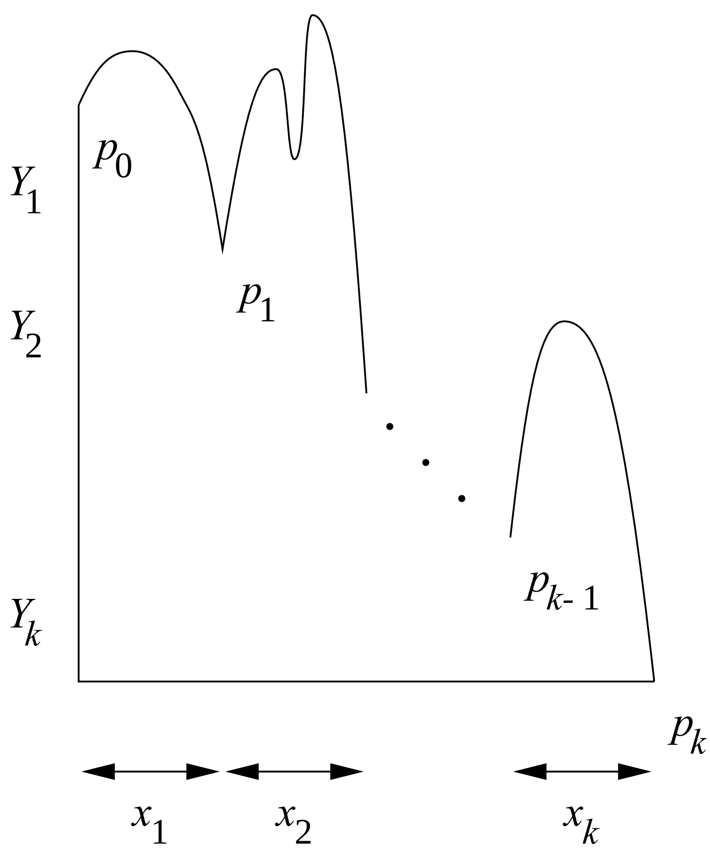
If Y_i is a terminal, we have $|x_i| = 1$, and $Y_i = x_i$. Thus $Y_i \overset{*}{\Rightarrow} x_i$ by the reflexivity of $\overset{*}{\Rightarrow}$.

Hence, $A \Longrightarrow Y_1 Y_2 \ldots Y_k \overset{*}{\Longrightarrow} x_1 x_2 \ldots x_k = x$

The claim of the theorem now follows by choosing $A = S$, and $x = w$. Suppose $w \in N(P)$. Then $(q, w, S) \overset{*}{\vdash} (q, \epsilon, \epsilon)$, and by (♣), we have $S \overset{*}{\Rightarrow} w$, meaning $w \in L(G)$.

From PDA's to CFG's

Let's look at how a PDA can consume $x = x_1x_2 \cdots x_k$ and empty the stack.



We shall define a grammar with variables of the form $[p_{i-1}Y_ip_i]$ representing going from p_{i-1} to p_i with net effect of popping Y_i.

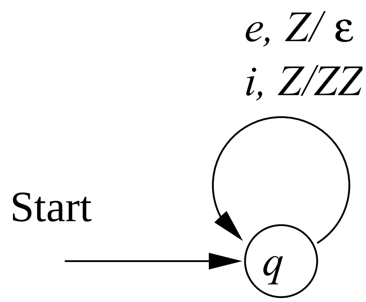
empty-stack

Formally, let $P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0)$ be a PDA. Define $G = (V, \Sigma, R, S)$, where

$$
\begin{aligned}
V &= \{[pXq] : \{p, q\} \subseteq Q, X \in \Gamma\} \cup \{S\} \\
R &= \{S \to [q_0 Z_0 p] : p \in Q\} \cup \\
&\quad \{[\mathbf{q} X r_k] \to a[\mathbf{r} Y_1 r_1] \cdots [r_{k-1} Y_k r_k] : \\
&\qquad a \in \Sigma \cup \{\epsilon\}, \\
&\qquad \{r_1, \ldots, r_k\} \subseteq Q, \\
&\qquad (\mathbf{r}, Y_1 Y_2 \cdots Y_k) \in \delta(\mathbf{q}, a, X)\}
\end{aligned}
$$

If $k = 0$, i.e. $Y_1 Y_2 \ldots Y_k = \varepsilon$, then [***q**X**r***] -> a

Example: Let's convert



$$P_N = (\{q\}, \{i, e\}, \{Z\}, \delta_N, q, Z),$$

where $\delta_N(q, i, Z) = \{(q, ZZ)\}$,
and $\delta_N(q, e, Z) = \{(q, \epsilon)\}$ to a grammar

$$G = (V, \{i, e\}, R, S),$$

where $V = \{[qZq], S\}$, and
$R = \{[qZq] \rightarrow i[qZq][qZq], [qZq] \rightarrow e,$ S -> [qZq] }

If we replace $[qZq]$ by A we get the productions
$S \rightarrow A$ and $A \rightarrow iAA|e$.

Example: Let $P = (\{p, q\}, \{0, 1\}, \{X, Z_0\}, \delta, q, Z_0)$, where δ is given by

1. $\delta(q, 1, Z_0) = \{(q, XZ_0)\}$

2. $\delta(q, 1, X) = \{(q, XX)\}$

3. $\delta(q, 0, X) = \{(p, X)\}$

4. $\delta(q, \epsilon, X) = \{(q, \epsilon)\}$

5. $\delta(p, 1, X) = \{(p, \epsilon)\}$

6. $\delta(p, 0, Z_0) = \{(q, Z_0)\}$

What language does this PDA accept?

to a CFG.

We get $G = (V, \{0, 1\}, R, S)$, where

$$V = \{[pXp], [pXq], [pZ_0p], [pZ_0q], S\}$$
$$[qXq], [pXq], [qZ_0p], [qZ_0q]$$

and the productions in R are

$S \to [qZ_0q] | [qZ_0p]$

From rule (1):

$[qZ_0q] \to 1[qXq][qZ_0q]$
$[qZ_0q] \to 1[qXp][pZ_0q]$
$[qZ_0p] \to 1[qXq][qZ_0p]$
$[qZ_0p] \to 1[qXp][pZ_0p]$

From rule (2):

$[qXq] \to 1[qXq][qXq]$
$[qXq] \to 1[qXp][pXq]$
$[qXp] \to 1[qXq][qXp]$
$[qXp] \to 1[qXp][pXp]$

From rule (3):

$[qXq] \rightarrow 0[pXq]$
$[qXp] \rightarrow 0[pXp]$

From rule (4):

$[qXq] \rightarrow \epsilon$

From rule (5):

$[pXp] \rightarrow 1$

From rule (6):

$[pZ_0q] \rightarrow 0[qZ_0q]$
$[pZ_0p] \rightarrow 0[qZ_0p]$

Theorem 6.14: Let G be constructed from a PDA P as above. Then $L(G) = N(P)$

Proof:

($\supseteq$-*direction.*) We shall show by an induction on the length of the sequence $\overset{*}{\vdash}$ that

(♠) If $(q, w, X) \overset{*}{\vdash} (p, \epsilon, \epsilon)$ then $[qXp] \overset{*}{\Rightarrow} w$.

Basis: Length 1. Then w is an a or ϵ, and $(p, \epsilon) \in \delta(q, w, X)$. By the construction of G we have $[qXp] \rightarrow w$ and thus $[qXp] \overset{*}{\Rightarrow} w$.

Induction: Length is $n > 1$, and ♠ holds for lengths $< n$. We must have

$$(q, w, X) \vdash (r_0, x, Y_1Y_2 \cdots Y_k) \vdash \cdots \vdash (p, \epsilon, \epsilon),$$

where $w = ax$ or $w = \epsilon x$. It follows that $(r_0, Y_1Y_2 \cdots Y_k) \in \delta(q, a, X)$. Then we have a production

$$[qXr_k] \rightarrow a[r_0Y_1r_1] \cdots [r_{k-1}Y_kr_k],$$

for all $\{r_1, \ldots, r_k\} \subset Q$.

We may now choose r_i to be the state in the sequence $\overset{*}{\vdash}$ when Y_i is popped. Let $x = w_1w_2 \cdots w_k$, where w_i is consumed while Y_i is popped. Then

$$(r_{i-1}, w_i, Y_i) \overset{*}{\vdash} (r_i, \epsilon, \epsilon).$$

Note that $r_k = p$

By the IH we get

$$[r_{i-1}, Y, r_i] \overset{*}{\Rightarrow} w_i$$

We then get the following derivation sequence:

$r_k = p$

$$[qXr_k] \Rightarrow a[r_0Y_1r_1]\cdots[r_{k-1}Y_kr_k] \overset{*}{\Rightarrow}$$

$$aw_1[r_1Y_2r_2][r_2Y_3r_3]\cdots[r_{k-1}Y_kr_k] \overset{*}{\Rightarrow}$$

$$aw_1w_2[r_2Y_3r_3]\cdots[r_{k-1}Y_kr_k] \overset{*}{\Rightarrow}$$

$$\cdots$$

$$aw_1w_2\cdots w_k = w \quad = ax$$

($\supseteq$-*direction.*) We shall show by an induction on the length of the derivation $\stackrel{*}{\Rightarrow}$ that

($\heartsuit$) If $[qXp] \stackrel{*}{\Rightarrow} w$ then $(q, w, X) \stackrel{*}{\vdash} (p, \epsilon, \epsilon)$

Basis: One step. Then we have a production $[qXp] \rightarrow w$. From the construction of G it follows that $(p, \epsilon) \in \delta(q, a, X)$, where $w = a$. But then $(q, w, X) \stackrel{*}{\vdash} (p, \epsilon, \epsilon)$.

Induction: Length of $\stackrel{*}{\Rightarrow}$ is $n > 1$, and $\heartsuit$ holds for lengths $< n$. Then we must have

$r_k = p$

$$[qXr_k] \Rightarrow a[r_0Y_1r_1][r_1Y_2r_2] \cdots [r_{k-1}Y_kr_k] \stackrel{*}{\Rightarrow} w$$

We can break w into $aw_1 \cdots w_k$ such that $[r_{i-1}Y_ir_i] \stackrel{*}{\Rightarrow} w_i$. From the IH we get

$$(r_{i-1}, w_i, Y_i) \stackrel{*}{\vdash} (r_i, \epsilon, \epsilon)$$

From Theorem 6.5 we get

$(r_{i-1}, w_i w_{i+1} \cdots w_k, Y_i Y_{i+1} \cdots Y_k) \vdash^*$
$(r_i, w_{i+1} \cdots w_k, Y_{i+1} \cdots Y_k)$

Since this holds for all $i \in \{1, \ldots, k\}$, we get

$(q, aw_1 w_2 \cdots w_k, X) \vdash$
$(r_0, w_1 w_2 \cdots w_k, Y_1 Y_2 \cdots Y_k) \vdash^*$
$(r_1, w_2 \cdots w_k, Y_2 \cdots Y_k) \vdash^*$
$(r_2, w_3 \cdots w_k, Y_3 \cdots Y_k) \vdash^*$
(p, ϵ, ϵ).

since $(r_0, Y_1 Y_2 \ldots Y_k)$ is in $\delta(q, a, X)$

$p = r_k$

Q1. Can you give a 1-state empty stack PDA for $L_1 = \{ 0^n 1^n \mid n >= 0\}$?

Q2: How to decide if a PDA M accepts a string w?

Deterministic PDA's

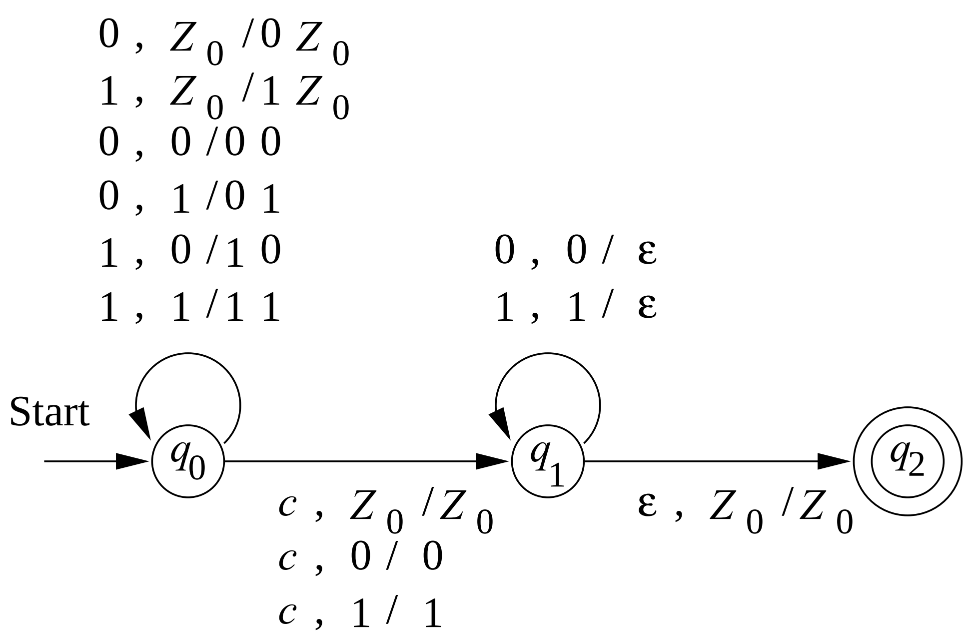
A PDA $P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ is *deterministic* iff

1. $\delta(q, \overset{\varepsilon}{a}, X)$ is always empty or a singleton.
2. If $\delta(q, a, X)$ is nonempty, then $\delta(q, \epsilon, X)$ must be empty.

Example: Let us define

$$L_{wcwr} = \{wcw^R : w \in \{0, 1\}^*\}$$

Then L_{wcwr} is recognized by the following DPDA



We'll show that Regular$\subset L$(DPDA) $\subset$ CFL

Theorem 6.17: If L is regular, then $L = L(P)$ for some DPDA P.

Proof: Since L is regular there is a DFA A s.t. $L = L(A)$. Let

$$A = (Q, \Sigma, \delta_A, q_0, F)$$

We define the DPDA

$$P = (Q, \Sigma, \{Z_0\}, \delta_P, q_0, Z_0, F),$$

where

$$\delta_P(q, a, Z_0) = \{(\delta_A(q, a), Z_0)\},$$

for all $p, q \in Q$, and $a \in \Sigma$.

An easy induction (do it!) on $|w|$ gives

$$(q_0, w, Z_0) \overset{*}{\vdash} (p, \epsilon, Z_0) \Leftrightarrow \hat{\delta_A}(q_0, w) = p$$

The theorem then follows (why?)

What about DPDA's that accept by null stack?

They can recognize only CFL's with the prefix property.

A language L has the *prefix property* if there are no two distinct strings in L, such that one is a prefix of the other.

Example: L_{wcwr} has the prefix property.

Example: $\{0\}^*$ does not have the prefix property.

Theorem 6.19: L is $N(P)$ for some DPDA P if and only if L has the prefix property and L is $L(P')$ for some DPDA P'.

Proof: Homework

- We have seen that Regular$\subseteq L$(DPDA).
- $L_{wcwr} \in L$(DPDA)$\setminus$ Regular
- Are there languages in CFL$\setminus L$(DPDA).

  Yes, for example L_{wwr}.

- What about DPDA's and Ambiguous Grammars?

  L_{wwr} has unamb. grammar $S \rightarrow 0S0|1S1|\epsilon$ but is not L(DPDA).

For the converse we have

Theorem 6.20: If $L = N(P)$ for some DPDA P, then L has an unambiguous CFG.

Proof: By inspecting the proof of Theorem 6.14 we see that if the construction is applied to a DPDA the result is a CFG with unique leftmost derivations.

Theorem 6.20 can actually be strengthen as follows

Theorem 6.21: If $L = L(P)$ for some DPDA P, then L has an unambiguous CFG.

Proof: Let \$ be a symbol outside the alphabet of L, and let $L' = L\$$.
It is easy to see that L' has the prefix property.
By Theorem 6.20 we have $L' = N(P')$ for some DPDA P'.
By Theorem 6.20 $N(P')$ can be generated by an unambiguous CFG G'
Modify G' into G, s.t. $L(G) = L$, by adding the production

$$\$ \rightarrow \epsilon$$

Since G' has unique leftmost derivations, G also has unique lm's, since the only new thing we're doing is adding derivations

$$w\$ \underset{lm}{\Rightarrow} w$$

to the end.