Closure Properties of Regular Languages

Let L and M be regular languages. Then the following languages are all regular:

- *Union:* $L \cup M$
- *Intersection:* $L \cap M$
- *Complement:* $\overline{N}$
- *Difference:* $L \setminus M$
- *Reversal:* $L^R = \{w^R : w \in L\}$
- *Closure:* L^*.
- *Concatenation:* $L.M$
- *Homomorphism:* $\boxed{h(a_1\ a_2 \ldots a_n) = h(a_1)h(a_2)\ldots h(a_n)}$

$h(L) = \{h(w) : w \in L, h$ is a homom. $\}$

- *Inverse homomorphism:*

$h^{-1}(L) = \{w \in \Sigma : h(w) \in L, h : \Sigma \rightarrow \Delta^*$ is a homom. $\}$

Theorem 4.4. For any regular L and M, $L \cup M$ is regular.

Proof. Let $L = L(E)$ and $M = L(F)$. Then $L(E + F) = L \cup M$ by definition.

Theorem 4.5. If L is a regular language over Σ, then so is $\overline{L} = \Sigma^* \setminus L$.

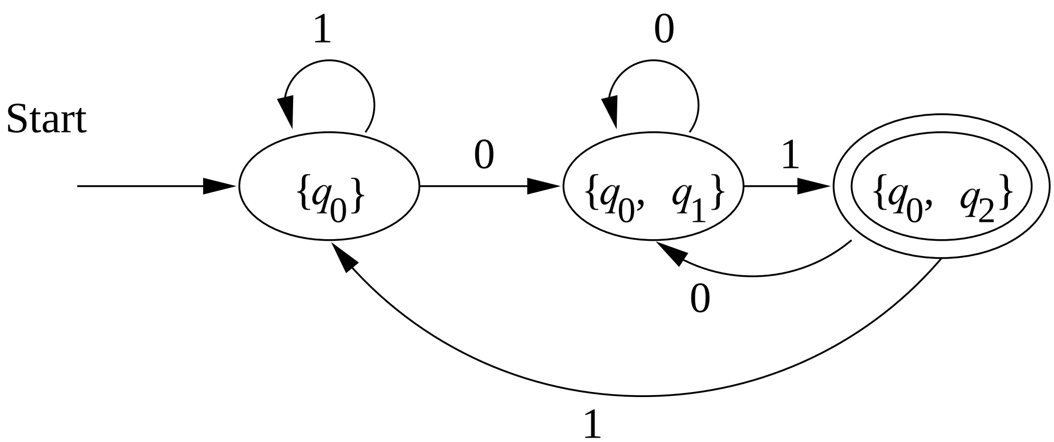
Proof. Let L be recognized by a DFA

$$A = (Q, \Sigma, \delta, q_0, F).$$

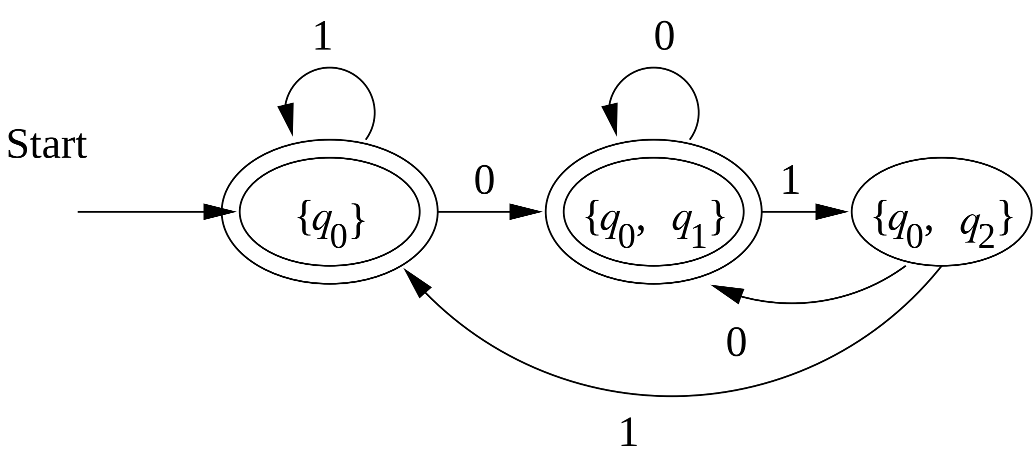
Let $B = (Q, \Sigma, \delta, q_0, Q \setminus F)$. Now $L(B) = \overline{L}$.

Example:

Let L be recognized by the DFA below



Then $\overline{L}$ is recognized by



Question: What are the regex's for L and $\overline{L}$

Theorem 4.8. If L and M are regular, then so is $L \cap M$.

Proof. By DeMorgan's law $L \cap M = \overline{\overline{L} \cup \overline{M}}$. We already that regular languages are closed under complement and union.

We shall shall also give a nice direct proof, the *Cartesian* construction from the e-commerce example.

Theorem 4.8. If L and M are regular, then so is $L \cap M$.

Proof. Let L be the language of

$$A_L = (Q_L, \Sigma, \delta_L, q_L, F_L)$$

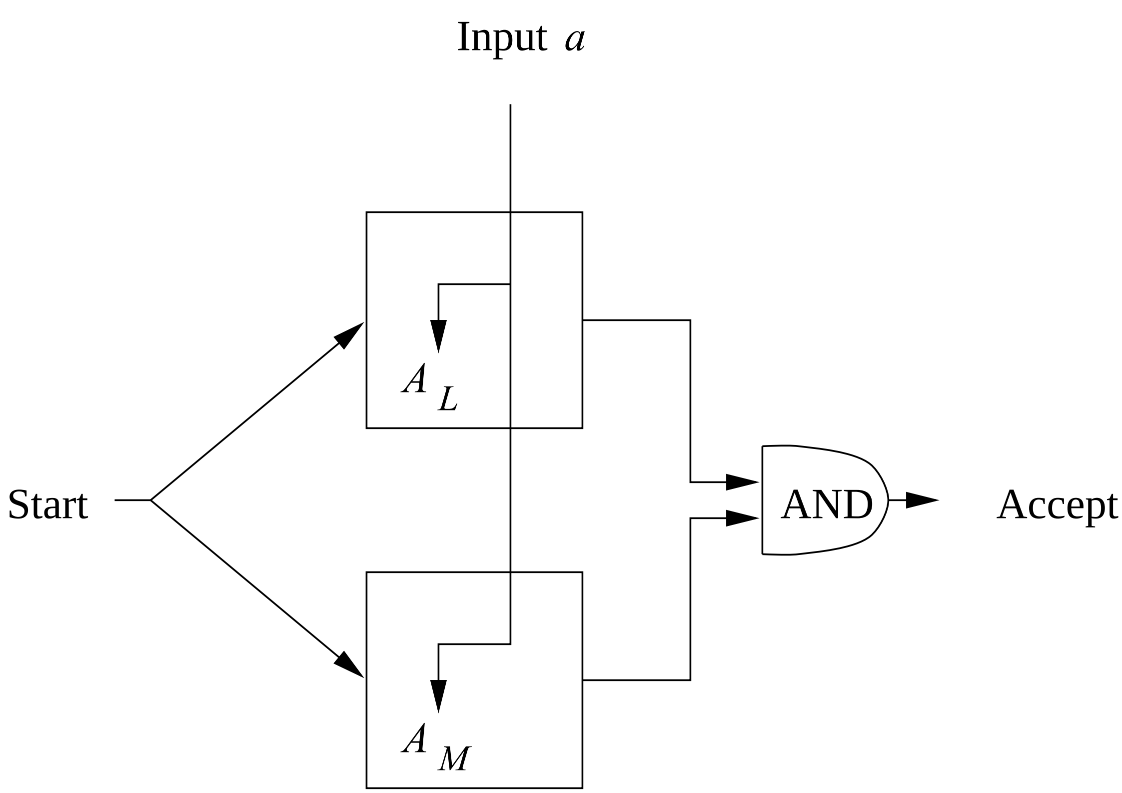
and M be the language of

$$A_M = (Q_M, \Sigma, \delta_M, q_M, F_M)$$

We assume w.l.o.g. that both automata are deterministic.

We shall construct an automaton that simulates A_L and A_M in parallel, and accepts if and only if both A_L and A_M accept.

If A_L goes from state p to state s on reading a, and A_M goes from state q to state t on reading a, then $A_{L \cap M}$ will go from state (p, q) to state (s, t) on reading a.



Formally

$$A_{L\cap M} = (Q_L \times Q_M, \Sigma, \delta_{L\cap M}, (q_L, q_M), F_L \times F_M),$$

where

$$\delta_{L\cap M}((p,q),a) = (\delta_L(p,a), \delta_M(q,a))$$

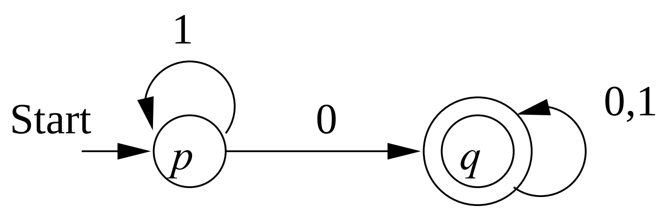
It will be shown in the tutorial by an induction on $|w|$ that

$$\hat{\delta}_{L\cap M}((q_L,q_M),w) = \left(\hat{\delta}_L(q_L,w), \hat{\delta}_M(q_M,w)\right)$$

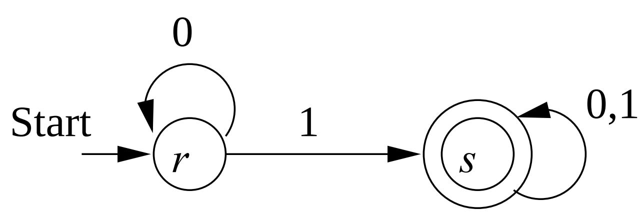
The claim then follows.

Question: Why?

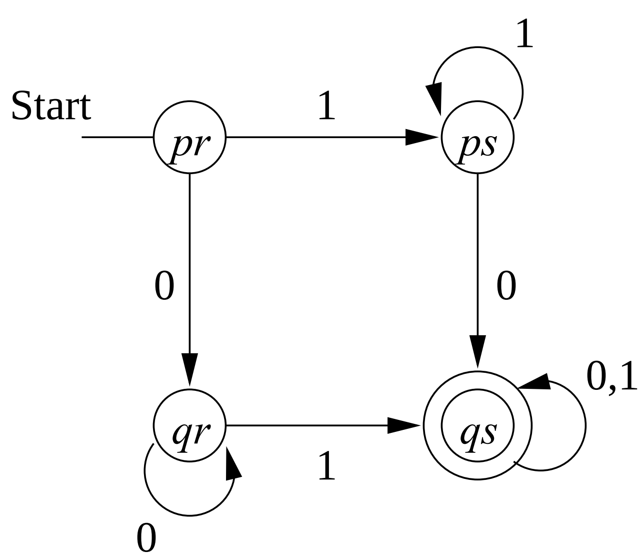
Example: $(c) = (a) \times (b)$



(a)



(b)



(c)

$\boxed{\text{Another example?}}$

Theorem 4.10. If L and M are regular languages, then so is $L \setminus M$.

Proof. Observe that $L \setminus M = L \cap \overline{M}$. We already know that regular languages are closed under complement and intersection.

Theorem 4.11. If L is a regular language, then so is L^R.

Proof 1: Let L be recognized by an FA A. Turn A into an FA for L^R, by

1. Reversing all arcs.

2. Make the old start state the new sole accepting state.

3. Create a new start state p_0, with $\delta(p_0, \epsilon) = F$ (the old accepting states).

Theorem 4.11. If L is a regular language, then so is L^R.

Proof 2: Let L be described by a regex E. We shall construct a regex E^R, such that $L(E^R) = (L(E))^R$.

We proceed by a structural induction on E.

Basis: If E is $\boldsymbol{\epsilon}$, $\emptyset$, or $\boldsymbol{a}$, then $E^R = E$.

Induction:

1. $E = F + G$. Then $E^R = F^R + G^R$
2. $E = F.G$. Then $E^R = G^R.F^R$
3. $E = F^*$. Then $E^R = (F^R)^*$

We will show by structural induction on E on blackboard in class that

$$L(E^R) = (L(E))^R$$

Homomorphisms

A *homomorphism* on Σ is a function $h : \Sigma \to \Theta^*$, where Σ and Θ are alphabets.

Let $w = a_1 a_2 \cdots a_n \in \Sigma^*$. Then

$$h(w) = h(a_1)h(a_2)\cdots h(a_n)$$

and

$$h(L) = \{h(w) : w \in L\}$$

Example: Let $h : \{0,1\}^* \to \{a,b\}^*$ be defined by $h(0) = ab$, and $h(1) = \epsilon$. Now $h(0011) = abab$.

Example: $h(L(\mathbf{10^*1})) = L((\boldsymbol{ab})^*)$.

Theorem 4.14: $h(L)$ is regular, whenever L is.

Proof: E.g., h(0*1+(0+1)*0) = h(0)*h(1)+(h(0)+h(1))*h(0)

Let $L = L(E)$ for a regex E. We claim that $L(h(E)) = h(L)$.

Basis: If E is ϵ or $\emptyset$. Then $h(E) = E$, and $L(h(E)) = L(E) = h(L(E))$.

If E is $\boldsymbol{a}$, then $L(E) = \{a\}$, $L(h(E)) = L(h(\boldsymbol{a})) = \{h(a)\} = h(L(E))$.

Induction:

Case 1: $G = E + F$. Now $L(h(E + F)) = L(h(E)+h(F)) = L(h(E)) \cup L(h(F)) = h(L(E)) \cup h(L(F)) = h(L(E) \cup L(F)) = h(L(E + F))$.

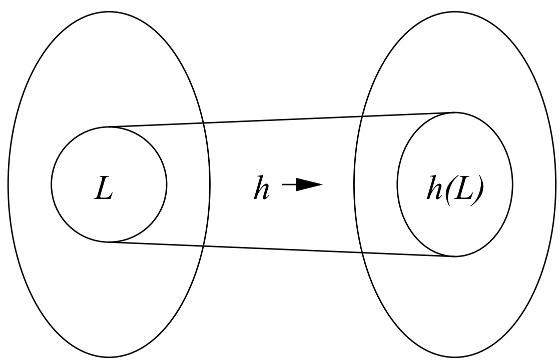
Case 2: $G = E.F$. Now $L(h(E.F)) = L(h(E)).L(h(F)) = h(L(E)).h(L(F)) = h(L(E).L(F)) = h(L(E.F))$

Case 3: $G = E^*$. Now $L(h(E^*)) = L(h(E)^*) = L(h(E))^* = h(L(E))^* = h(L(E)^*) = h(L(E^*))$

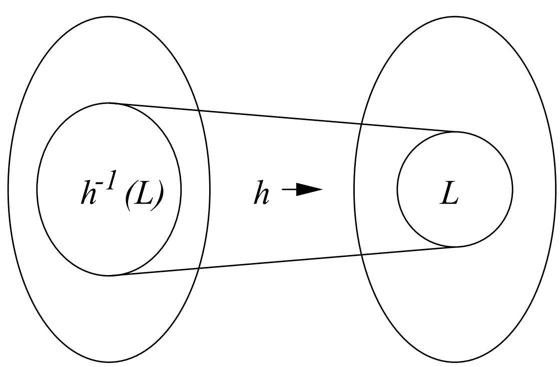
Inverse Homomorphism

Let $h : \Sigma \to \Theta^*$ be a homom. Let $L \subseteq \Theta^*$, and define

$$h^{-1}(L) = \{w \in \Sigma^* : h(w) \in L\}$$



(a)



(b)

Example: Let $h : \{a, b\} \to \{0, 1\}^*$ be defined by $h(a) = 01$, and $h(b) = 10$. If $L = L((\mathbf{00} + \mathbf{1})^*)$, then $h^{-1}(L) = L((\boldsymbol{ba})^*)$.

Claim: $h(w) \in L$ if and only if $w = (ba)^n$

Proof: Let $w = (ba)^n$. Then $h(w) = (1001)^n \in L$.

Let $h(w) \in L$, and suppose $w \notin L((\boldsymbol{ba})^*)$. There are four cases to consider.

1. w begins with a. Then $h(w)$ begins with 01 and $\notin L((\mathbf{00} + \mathbf{1})^*)$.

2. w ends in b. Then $h(w)$ ends in 10 and $\notin L((\mathbf{00} + \mathbf{1})^*)$.

3. $w = xaay$. Then $h(w) = z0101v$ and $\notin L((\mathbf{00} + \mathbf{1})^*)$.

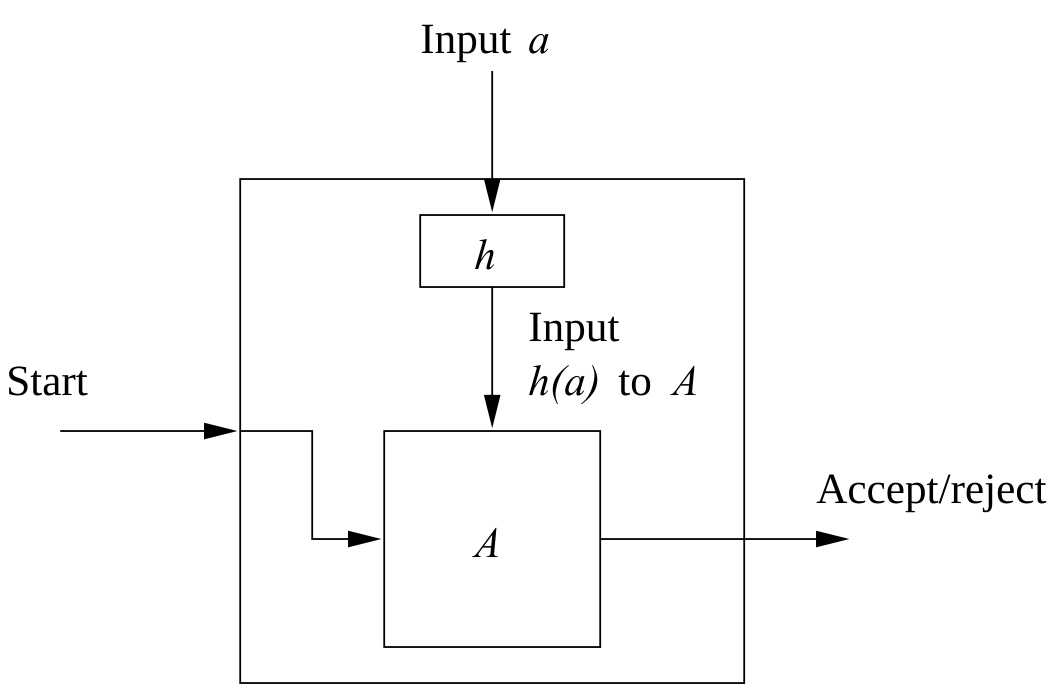
4. $w = xbby$. Then $h(w) = z1010v$ and $\notin L((\mathbf{00} + \mathbf{1})^*)$.

Theorem 4.16: Let $h : \Sigma \rightarrow \Theta^*$ be a homom., and $L \subseteq \Theta^*$ regular. Then $h^{-1}(L)$ is regular.

Proof: Let L be the language of $A = (Q, \Theta, \delta, q_0, F)$. We define $B = (Q, \Sigma, \gamma, q_0, F)$, where

$$\gamma(q, a) = \widehat{\delta}(q, h(a))$$

It will be shown by induction on $|w|$ in the tutorial that $\widehat{\gamma}(q_0, w) = \widehat{\delta}(q_0, h(w))$



Decision Properties

We consider the following:

1. Converting among representations for regular languages.

2. Is $L = \emptyset$?

3. Is $w \in L$?

4. Do two descriptions define the same language?

From NFA's to DFA's

Suppose the ϵ-NFA has n states.

To compute $\text{ECLOSE}(p)$ we follow at most n^2 arcs.

The DFA has 2^n states, for each state S and each $a \in \Sigma$ we compute $\delta_D(S, a)$ in n^3 steps. Grand total is $O(n^3 2^n)$ steps.

If we compute δ for reachable states only, we need to compute $\delta_D(S, a)$ only s times, where s is the number of reachable states. Grand total is $O(n^3 s)$ steps.

From DFA to NFA

All we need to do is to put set brackets around the states. Total $O(n)$ steps.

From FA to regex

We need to compute n^3 entries of size up to 4^n. Total is $O(n^3 4^n)$.

The FA is allowed to be a NFA. If we first wanted to convert the NFA to a DFA, the total time would be doubly exponential

From regex to FA's We can build an expression tree for the regex in n steps.

We can construct the automaton in n steps.

Eliminating ϵ-transitions takes $O(n^3)$ steps.

If you want a DFA, you might need an exponential number of steps.

Testing emptiness

$L(A) \neq \emptyset$ for FA A if and only if a final state is reachable from the start state in A. Total $O(n^2)$ steps.

Alternatively, we can inspect a regex E and tell if $L(E) = \emptyset$. We use the following method:

$E = F + G$. Now $L(E)$ is empty if and only if both $L(F)$ and $L(G)$ are empty.

$E = F.G$. Now $L(E)$ is empty if and only if either $L(F)$ or $L(G)$ is empty.

$E = F^*$. Now $L(E)$ is never empty, since $\epsilon \in L(E)$.

$E = \epsilon$. Now $L(E)$ is not empty.

$E = \boldsymbol{a}$. Now $L(E)$ is not empty.

$E = \boldsymbol{\emptyset}$. Now $L(E)$ is empty.

Testing membership

To test $w \in L(A)$ for DFA A, simulate A on w. If $|w| = n$, this takes $O(n)$ steps.

If A is an NFA and has s states, simulating A on w takes $O(ns^2)$ steps.

If A is an ϵ-NFA and has s states, simulating A on w takes $O(ns^3)$ steps.

If $L = L(E)$, for regex E of length s, we first convert E to an ϵ-NFA with $2s$ states. Then we simulate w on this machine, in $O(ns^3)$ steps.

Does L((0+1)*0(0+1)^{3}1*) contain 10101011 or 101011101?

Finiteness: How to decide if L(A) is finite for DFA A?

Equivalence and Minimization of Automata

Let $A = (Q, \Sigma, \delta, q_0, F)$ be a DFA, and $\{p, q\} \subseteq Q$. We define

$$p \equiv q \;\Leftrightarrow\; \forall w \in \Sigma^* \;:\; \hat{\delta}(p, w) \in F \text{ iff } \hat{\delta}(q, w) \in F$$

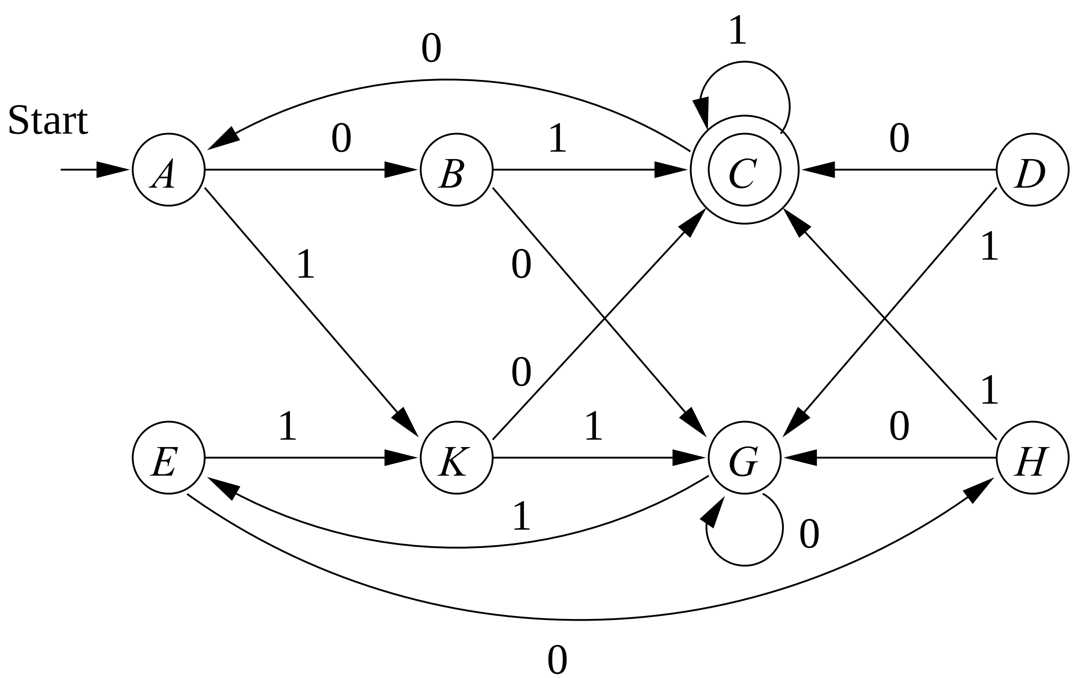
- If $p \equiv q$ we say that p and q are *equivalent*

- If $p \not\equiv q$ we say that p and q are *distinguishable*

IOW (in other words) p and q are distinguishable iff

$\exists w : \hat{\delta}(p, w) \in F$ and $\hat{\delta}(q, w) \notin F$, or vice versa

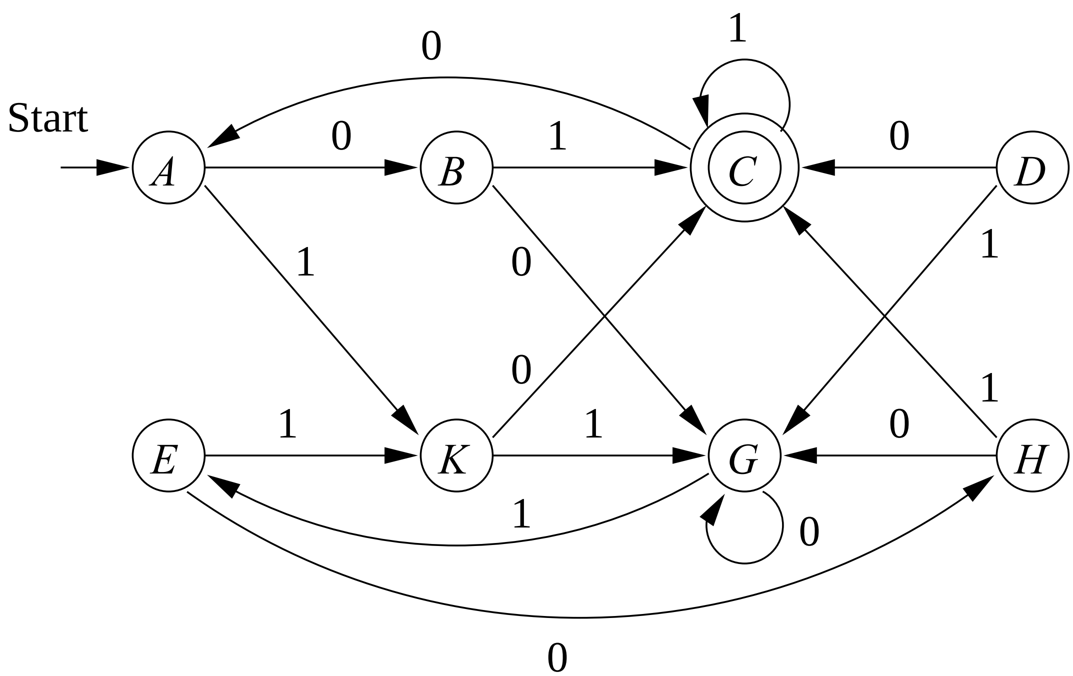
Example:



$$\widehat{\delta}(C,\epsilon) \in F, \widehat{\delta}(G,\epsilon) \notin F \Rightarrow C \not\equiv G$$

$$\widehat{\delta}(A,01) = C \in F, \widehat{\delta}(G,01) = E \notin F \Rightarrow A \not\equiv G$$

What about A and E?



$\widehat{\delta}(A, \epsilon) = A \notin F, \widehat{\delta}(E, \epsilon) = E \notin F$

$\widehat{\delta}(A, 1) = \mathrm{K} = \widehat{\delta}(E, 1)$

Therefore $\widehat{\delta}(A, 1x) = \widehat{\delta}(E, 1x) = \widehat{\delta}(\mathrm{K}, x)$

$\widehat{\delta}(A, 00) = G = \widehat{\delta}(E, 00)$

$\widehat{\delta}(A, 01) = C = \widehat{\delta}(E, 01)$

Conclusion: $A \equiv E$.

We can compute distinguishable pairs with the following inductive *table filling algorithm*:

Basis: If $p \in F$ and $q \notin F$, then $p \not\equiv q$.

Induction: If $\exists a \in \Sigma : \delta(p, a) \not\equiv \delta(q, a)$,
then $p \not\equiv q$.

Example: Applying the table filling algo to A:

	A	B	C	D	E	K	G
B	x						
C	x	x					
D	x	x	x				
E		x	x	x			
K	x	x	x		x		
G	x	x	x	x	x	x	
H	x		x	x	x	x	x

Theorem 4.20: If p and q are not distinguished by the TF-algo, then $p \equiv q$.

Proof: Suppose to the contrary that that there is a *bad pair* $\{p, q\}$, s.t.

1. $\exists w : \widehat{\delta}(p, w) \in F, \widehat{\delta}(q, w) \notin F$, or vice versa.

2. The TF-algo does not distinguish between p and q.

Let $w = a_1 a_2 \cdots a_n$ be the shortest string that identifies a bad pair $\{p, q\}$.

Now $w \neq \epsilon$ since otherwise the TF-algo would in the basis distinguish p from q. Thus $n \geq 1$.

Consider states $r = \delta(p, a_1)$ and $s = \delta(q, a_1)$. Now $\{r, s\}$ cannot be a bad pair since $\{r, s\}$ would be indentified by a string shorter than w. Therefore, the TF-algo must have discovered that r and s are distinguishable.

But then the TF-algo would distinguish p from q in the inductive part.

Thus there are no bad pairs and the theorem is true.

Testing Equivalence of Regular Languages

Let L and M be reg langs (each given in some form).

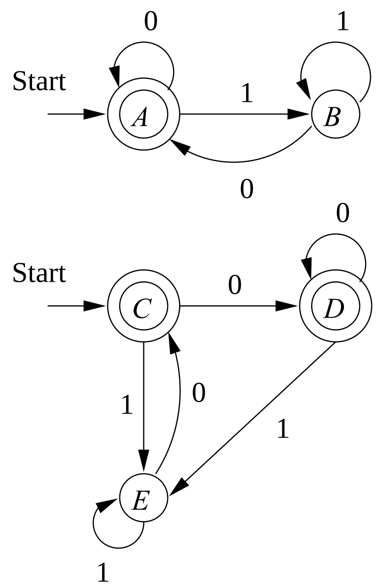
To test if $L = M$

1. Convert both L and M to DFA's.

2. Imagine a DFA that is the union of the two DFA's (never mind there are two start states)

3. If TF-algo says that the two start states are distinguishable, then $L \neq M$, otherwise $L = M$.

Example:



We can "see" that both DFA accept $L(\epsilon + (\mathbf{0}+\mathbf{1})^*\mathbf{0})$. The result of the TF-algo is

	A	B	C	D
B	x			
C		x		
D		x		
E	x		x	x

Therefore the two automata are equivalent.

Minimization of DFA's

We can use the TF-algo to minimize a DFA by merging all equivalent states. IOW, replace each state p by $p/_\equiv$.

Example: The DFA on slide 119 has equivalence classes $\{\{A, E\}, \{B, H\}, \{C\}, \{D, K\}, \{G\}\}$.

The "union" DFA on slide 125 has equivalence classes $\{\{A, C, D\}, \{B, E\}\}$.

Note: In order for $p/_\equiv$ to be an *equivalence class*, the relation $\equiv$ has to be an *equivalence relation* (reflexive, symmetric, and transitive).

Theorem 4.23: If $p \equiv q$ and $q \equiv r$, then $p \equiv r$.

Proof: Suppose to the contrary that $p \not\equiv r$. Then $\exists w$ such that $\hat{\delta}(p, w) \in F$ and $\hat{\delta}(r, w) \not\in F$, or vice versa.

OTH, $\hat{\delta}(q, w)$ is either accpeting or not.

Case 1: $\hat{\delta}(q, w)$ is accepting. Then $q \not\equiv r$.

Case 2: $\hat{\delta}(q, w)$ is not accepting. Then $p \not\equiv q$.

The vice versa case is proved symmetrically

Therefore it must be that $p \equiv r$.

Assume A has no inaccessible states.

To minimize a DFA $A = (Q, \Sigma, \delta, q_0, F)$ construct a DFA $B = (Q/_{\equiv}, \Sigma, \gamma, q_0/_{\equiv}, F/_{\equiv})$, where

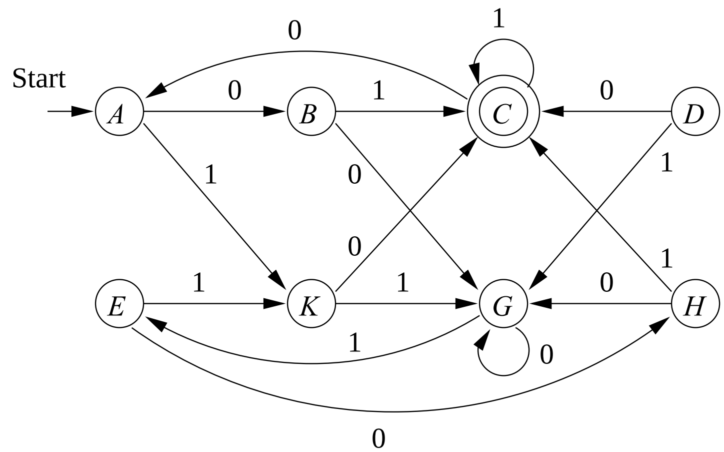
$$\gamma(p/_{\equiv}, a) = \delta(p, a)/_{\equiv}$$

In order for B to be well defined we have to show that

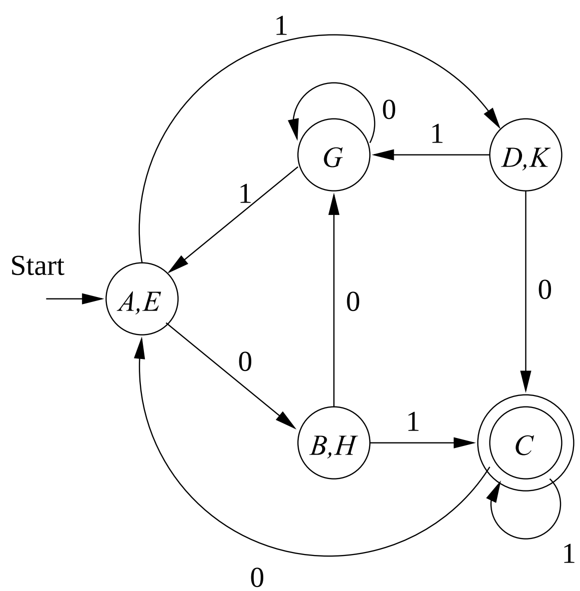
$$\text{If } p \equiv q \text{ then } \delta(p, a) \equiv \delta(q, a)$$

If $\delta(p, a) \not\equiv \delta(q, a)$, then the TF-algo would conclude $p \not\equiv q$, so B is indeed well defined. Note also that $F/_{\equiv}$ contains all and only the accepting states of A.

Example: We can minimize

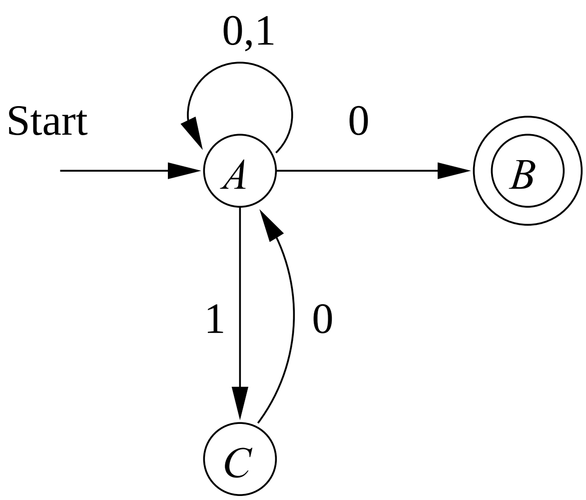


to obtain



NOTE: We cannot apply the TF-algo to NFA's.

For example, to minimize



we simply remove state C.

However, $A \not\equiv C$.

Why the Minimized DFA Can't Be Beaten

Let B be the minimized DFA obtained by applying the TF-algo to DFA A.

We already know that $L(A) = L(B)$.

What if there existed a DFA C, with
$L(C) = L(B)$ and fewer states than B?

Then run the TF-algo on B "union" C.

Since $L(B) = L(C)$ we have $q_0^B \equiv q_0^C$.

Also, $\delta(q_0^B, a) \equiv \delta(q_0^C, a)$, for any a.

Claim: For each state p in B there is at least one state q in C, s.t. $p \equiv q$.

Proof of claim: There are no inaccessible states, so $p = \widehat{\delta}(q_0^B, a_1 a_2 \cdots a_k)$, for some string $a_1 a_2 \cdots a_k$. Now $q = \widehat{\delta}(q_0^C, a_1 a_2 \cdots a_k)$, and $p \equiv q$.

Since C has fewer states than B, there must be two states r and s of B such that $r \equiv t \equiv s$, for some state t of C. But then $r \equiv s$ (why?) which is a contradiction, since B was constructed by the TF-algo.