Name:

SID:

**QUESTION 1.** [10 pts] Design a DFA to accept the following language:

\[ L = \{ x \mid x \in \{0,1\}^*, \text{ the number of 0's in } x \text{ is divisible by 4 and the number of 1's in } x \text{ is divisible by 2} \} \]

**Answer:**

\[
\begin{array}{c|cc}
   \quad & 0 & 1 \\
* \rightarrow & q_0.0 & q_1.0 & q_0.1 \\
q_0.1 & q_1.1 & q_0.0 \\
q_1.0 & q_2.0 & q_1.1 \\
q_1.1 & q_2.1 & q_1.0 \\
q_2.0 & q_3.0 & q_2.1 \\
q_2.1 & q_3.1 & q_2.0 \\
q_3.0 & q_0.0 & q_3.1 \\
q_3.1 & q_0.1 & q_3.0 \\
\end{array}
\]

Give partial credits for DFAs that handles the number of 0’s or the number of 1’s correctly.
QUESTION 2. [10 pts] Design an ε-NFA to accept the following language:

\[ L = \{ x \mid x \in \{0, 1\}^*, \ x \text{ begins or ends with 001} \} \]

Answer:

Note that the answer is not unique.
**QUESTION 3.** [10 pts] Convert the following NFA to a DFA:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rightarrow q_0$</td>
<td>${q_1}$</td>
<td>${q_0, q_1}$</td>
</tr>
<tr>
<td>$q_1$</td>
<td>${q_1, q_2}$</td>
<td>${q_2}$</td>
</tr>
<tr>
<td>$^*q_2$</td>
<td>${q_1}$</td>
<td>$\emptyset$</td>
</tr>
</tbody>
</table>

Answer:

<table>
<thead>
<tr>
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<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
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<td>${q_1}$</td>
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</tr>
<tr>
<td>${q_1}$</td>
<td>${q_1, q_2}$</td>
<td>${q_2}$</td>
</tr>
<tr>
<td>$^*{q_2}$</td>
<td>${q_1}$</td>
<td>$\emptyset$</td>
</tr>
<tr>
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<td>${q_1, q_2}$</td>
<td>${q_0, q_1, q_2}$</td>
</tr>
<tr>
<td>$^*{q_1, q_2}$</td>
<td>${q_1, q_2}$</td>
<td>${q_2}$</td>
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<td>${q_0, q_1, q_2}$</td>
</tr>
<tr>
<td>$\emptyset$</td>
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</tr>
</tbody>
</table>

It’s okay to include inaccessible states (i.e., $\{q_0, q_2\}$). Give partial credits for correct steps.
QUESTION 4. [10 pts] Give a regular expression for the following language:

\[ L = \left\{ x \mid x \in \{0,1\}^*, \ x \text{ has at most two } 1\text{'s between consecutive } 0\text{'s} \right\} \]

For example, the language constains strings 111, 10111, 11011010010111, but not strings like 1011110 or 01110100.

Answer:

\[ 1^* + 1^*0((\epsilon + 1 + 11)0)^*1^* \]

The regex is not unique. Give partial credits for regex’s containing some correct components.
QUESTION 5. [10 pts] Convert the following DFA to a regular expression by eliminating its states in the order: \( q_2, q_1 \).

<table>
<thead>
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<th></th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rightarrow *q_0 )</td>
<td>( q_2 )</td>
<td>( q_1 )</td>
</tr>
<tr>
<td>( q_1 )</td>
<td>( q_2 )</td>
<td>( q_0 )</td>
</tr>
<tr>
<td>( q_2 )</td>
<td>( q_1 )</td>
<td>( q_2 )</td>
</tr>
</tbody>
</table>

Answer:

After eliminating state \( q_2 \), we have a new transition from \( q_1 \) to \( q_1 \) with label \( 01^*0 \) (2 pts), and the arc label from \( q_0 \) to \( q_1 \) becomes \( 1 + 01^*0 \) (2 pts).

After eliminating state \( q_1 \), the arc label from \( q_0 \) to \( q_0 \) becomes (3 pts)

\[(1 + 01^*0)(01^*0)^*1\]

which leaves the final answer as (3 pts)

\[((1 + 01^*0)(01^*0)^*1)^*\]

Give partial credits for correct steps. Deduct at least 3 pts for eliminating states in a different order.
QUESTION 6. [15 pts] Using the test technique learned in class, prove or disprove the following general identities (or algebraic laws) concerning regular expressions:

1. \((E + F)^* = E^* + F^*\)
2. \((E + \epsilon)^* = E^*\)

Answer:

1. False (1 pt). Replace \(E\) by symbol \(a\) and \(F\) by symbol \(b\). Then LHS = \((ab)^*\) and RHS = \(a^* + b^*\) (2 pts). They are clearly not equal, because the string \(ab\) is in \(L((ab)^*)\) but not in \(L(a^* + b^*)\). (3 pts)

2. True (1 pt). Replace \(E\) by symbol \(a\). Then LHS = \((a + \epsilon)^*\) and RHS = \(a^*\) (2 pts). Since \(L(a^*)\) is the universe over the alphabet \(\{a\}\), clearly \(L((a + \epsilon)^*) \subseteq L(a^*)\) (3 pts). On the other hand, since \(L(a) \subseteq L(a + \epsilon)\), \(L(a^*) \subseteq L((a + \epsilon)^*)\) (3 pts). Hence, \((a + \epsilon)^* = a^*\).

Give partial credits for major correct ideas/steps.
QUESTION 7. [10 pts] Prove that the following language is not regular using the Pumping Lemma:

\[ L = \{0^i1^j2^{2i} \mid i, j \geq 1\} \]

Answer:

Let \( n \) be the constant in the Pumping Lemma. Pick \( w = 0^n12^{2n} \in L \). Let \( w = xyz \) be any partition satisfying (i) \(|y| > 0\) and \(|xy| \leq n\). Clearly, \( x = 0^s \) and \( y = 0^t \) for some \( t > 0 \). Then \( xyyz = 0^{n+t}12^{2n} \notin L \), and thus a contradiction.

Give partial credits for correct/reasonable steps.