QUESTION 1. [10 pts] Design a DFA to accept the following language:

\[ L = \{ x \mid x \in \{0,1\}^*, \text{ the number of } 0\text{'s in } x \text{ is divisible by 4 and the number of } 1\text{'s in } x \text{ is divisible by 2} \} \]

Answer:

\[
\begin{array}{c|cc}
& 0 & 1 \\
\hline
\ast & q_0,0 & q_1,0 & q_0,1 \\
q_0,1 & q_1,1 & q_0,0 \\
q_1,0 & q_2,0 & q_1,1 \\
q_1,1 & q_2,1 & q_1,0 \\
q_2,0 & q_3,0 & q_2,1 \\
q_2,1 & q_3,1 & q_2,0 \\
q_3,0 & q_0,0 & q_3,1 \\
q_3,1 & q_0,1 & q_3,0 \\
\end{array}
\]

Give partial credits for DFAs that handles the number of 0’s or the number of 1’s correctly.
QUESTION 2. [10 pts] Design an $\epsilon$-NFA to accept the following language:

$$L = \{x \mid x \in \{0, 1\}^*, x \text{ begins or ends with } 101\}$$

Answer:

<table>
<thead>
<tr>
<th></th>
<th>$\epsilon$</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q_0$</td>
<td>${q_1, q_2}$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$q_1$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>${q_3}$</td>
</tr>
<tr>
<td>$q_3$</td>
<td>$\emptyset$</td>
<td>${q_5}$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$q_5$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>${q_7}$</td>
</tr>
<tr>
<td>$q_7$</td>
<td>$\emptyset$</td>
<td>${q_7}$</td>
<td>${q_7}$</td>
</tr>
<tr>
<td>$q_2$</td>
<td>$\emptyset$</td>
<td>${q_2}$</td>
<td>${q_2}$</td>
</tr>
<tr>
<td>$q_4$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>${q_1}$</td>
</tr>
<tr>
<td>$q_6$</td>
<td>$\emptyset$</td>
<td>${q_6}$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$q_8$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
</tr>
</tbody>
</table>

Note that the answer is not unique.
QUESTION 3. [10 pts] Convert the following NFA to a DFA:

\[
\begin{array}{c|c|c}
0 & 1 \\
\hline
q_0 & \{q_1\} & \{q_0, q_1\} \\
q_1 & \{q_1, q_2\} & \{q_2\} \\
q_2 & \{q_1\} & \emptyset \\
\end{array}
\]

Answer:

\[
\begin{array}{c|c|c}
0 & 1 \\
\hline
\{q_0\} & \{q_1\} & \{q_0, q_1\} \\
\{q_1\} & \{q_1, q_2\} & \{q_2\} \\
*\{q_2\} & \{q_1\} & \emptyset \\
\{q_0, q_1\} & \{q_1, q_2\} & \{q_0, q_1, q_2\} \\
*\{q_1, q_2\} & \{q_1, q_2\} & \{q_2\} \\
*\{q_0, q_1, q_2\} & \{q_1, q_2\} & \{q_0, q_1, q_2\} \\
\emptyset & \emptyset & \emptyset \\
\end{array}
\]

It’s okay to include inaccessible states (i.e., \{q_0, q_2\}). Give partial credits for correct steps.
QUESTION 4. [10 pts] Give a regular expression for the following language:

$L = \{ x \mid x \in \{0,1\}^*, \text{ } x \text{ has at most two 0's between consecutive 1's}\}$

For example, the language contains strings $000, 01000, 00100101101000$, but not strings like $010001$ or $10001011$.

Answer:

$$0^* + 0^*1((\epsilon + 0 + 00)1)^*0^*$$

The regex is not unique. Give partial credits for regex’s containing some correct components.
QUESTION 5. [10 pts] Prove or disprove the following identities. Note that, you can disprove an identity by means of a counterexample.

1. $(0^*1^*)^* = (0^*1^*)^* + (01^*)^*$
2. $1(01)^* = (10)^*1$

Answer:

1. False (2 pts). E.g., $10$ (2 pts).
2. True (2 pts). Consider any string $w \in L(1(01)^*)$. Clearly, $w = 1(01)^n$ for some integer $n \geq 0$. But, such a string can also be rewritten as $w = (10)^n1$. Hence, $w \in L((10)^*1)$ and $L(1(01)^*) \subseteq L((10)^*1)$. Similarly, we can prove that $L((10)^*1) \subseteq L(1(01)^*)$. (4 pts)