CS 150 Lecture Slides

Motivation

- Automata = abstract computing devices

- Turing studied Turing Machines (= computers) before there were any real computers

- We will also look at simpler devices than Turing machines (Finite Automata, Pushdown Automata, . . .), and specification means, such as grammars and regular expressions.

Note specification is also computation!

- Unsolvability/undecidability/uncomputability = what cannot be computed by algorithms

Finite Automata

Finite Automata are used as a model for

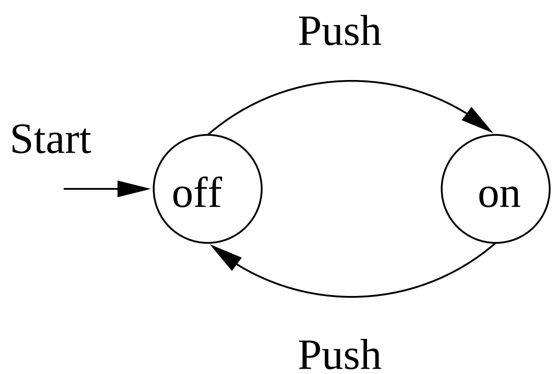
- Software for designing digital circuits
- Lexical analyzer of a compiler
- Searching for keywords in a file or on the web

Automata-based programming!

- Software for verifying finite state systems, such as communication protocols

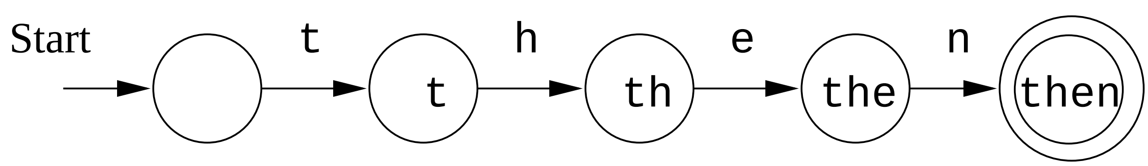
- Computer security
- Computer graphics and fractal compression
- Machine learning/NLP
- Virtual currency (block chains)

• Example: Finite Automaton modelling an on/off switch



Model of Computation:
A program

• Example: Finite Automaton recognizing the string **`then`**



Model of Description:
A specification

Structural Representations

These are alternative ways of describing an abstract model.

Grammars: A rule like $E \Rightarrow E + E$ specifies an arithmetic expression

- $Lineup \Rightarrow Person.Lineup$

Recursion!

says that a lineup is a person in front of a lineup.

Regular Expressions: Denote structure of data, e.g.

What symbols are allowed, how are they ordered, what can be repeated, etc.

`'[A-Z][a-z]*[][A-Z][A-Z]'`

matches `Ithaca NY`

does not match `Palo Alto CA`

Question: What expression would match `Palo Alto CA`

[A-Z][a-z]*[ ][A-Z][a-z]*[ ][A-Z][A-Z]

Central Concepts

Alphabet: Finite, nonempty set of symbols

Example: $\Sigma = \{0, 1\}$ binary alphabet

Example: $\Sigma = \{a, b, c, \ldots, z\}$ the set of all lower case letters

Example: The set of all ASCII characters

Strings: Finite sequence of symbols from an alphabet Σ, e.g. 0011001

Empty String: The string with zero occurrences of symbols from Σ

- The empty string is denoted ϵ

Length of String: Number of positions for symbols in the string.

$|w|$ denotes the length of string w

$|0110| = 4, |\epsilon| = 0$

Powers of an Alphabet: Σ^k = the set of strings of length k with symbols from Σ

Example: $\Sigma = \{0, 1\}$

$\Sigma^1 = \{0, 1\}$

$\Sigma^2 = \{00, 01, 10, 11\}$

$\Sigma^0 = \{\epsilon\}$

Question: How many strings are there in Σ^3

The set of all strings over Σ is denoted Σ^*

$\Sigma^* = \Sigma^0 \cup \Sigma^1 \cup \Sigma^2 \cup \cdots$

E.g. {0,1}*

the universe of {0,1}

Also:

$\Sigma^+ = \Sigma^1 \cup \Sigma^2 \cup \Sigma^3 \cup \cdots$

$\Sigma^* = \Sigma^+ \cup \{\epsilon\}$

Concatenation: If x and y are strings, then xy is the string obtained by placing a copy of y immediately after a copy of x

$x = a_1 a_2 \ldots a_i, y = b_1 b_2 \ldots b_j$

$xy = a_1 a_2 \ldots a_i b_1 b_2 \ldots b_j$

Example: $x = 01101, y = 110, xy = 01101110$

Note: For any string x

$$x\epsilon = \epsilon x = x$$

Languages:

If Σ is an alphabet, and $L \subseteq \Sigma^*$
then L is a language

Examples of languages:

- The set of legal English words

- The set of legal C programs

- The set of strings consisting of n 0's followed by n 1's

$$\{\epsilon, 01, 0011, 000111, \ldots\}$$

$$\{0^n 1^n | n >= 0\}$$

- The set of strings with equal number of 0's and 1's

$$\{\epsilon, 01, 10, 0011, 0101, 1001, \ldots\}$$

- $L_P =$ the set of binary numbers whose value is prime

$$\{10, 11, 101, 111, 1011, \ldots\}$$

- The empty language $\emptyset$

- The language $\{\epsilon\}$ consisting of the empty string

Note: $\emptyset \neq \{\epsilon\}$

Note2: The underlying alphabet Σ is always finite

Problem: Is a given string w a member of a language L? (Membership Question)

Example: Is a binary number prime = is it a member in L_P

Is $11101 \in L_P$? What computational resources are needed to answer the question.

Usually we think of problems not as a yes/no decision, but as something that transforms an input into an output.

Example: Parse a C-program = check if the program is correct, and if it is, produce a parse tree.

Let L_X be the set of all valid programs in prog lang X. If we can show that determining membership in L_X is hard, then parsing programs written in X cannot be easier.

Question: Why?

L => membership question of L => decision problem
membership question of L <= L <=

language == (decision) problem!

Quick Quiz

Is the following
   a) an alphabet
   b) a string
   c) and/or a language
(multiple answers are allowed)

1) 0101001
2) {01, 11, 101, 111, 1011}
3) {0, 1, 2}
4) {ε, 0, 1}

Finite Automata Informally

Protocol for e-commerce using e-money

Allowed events:

1. The customer can *pay* the store (=send the money-file to the store)

2. The customer can *cancel* the money (like putting a stop on a check)

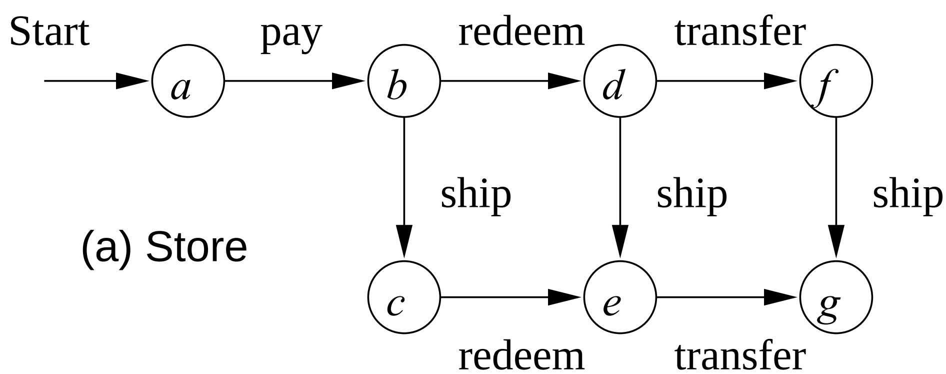
3. The store can *ship* the goods to the customer

4. The store can *redeem* the money (=cash the check)

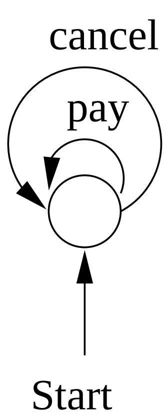
5. The bank can *transfer* the money to the store

e-commerce

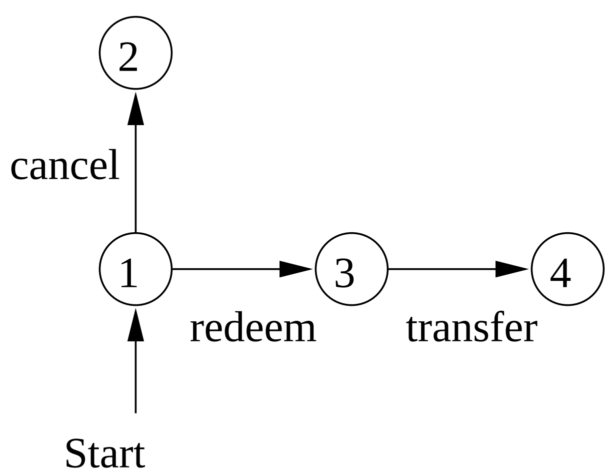
The protocol for each participant:



(a) Store

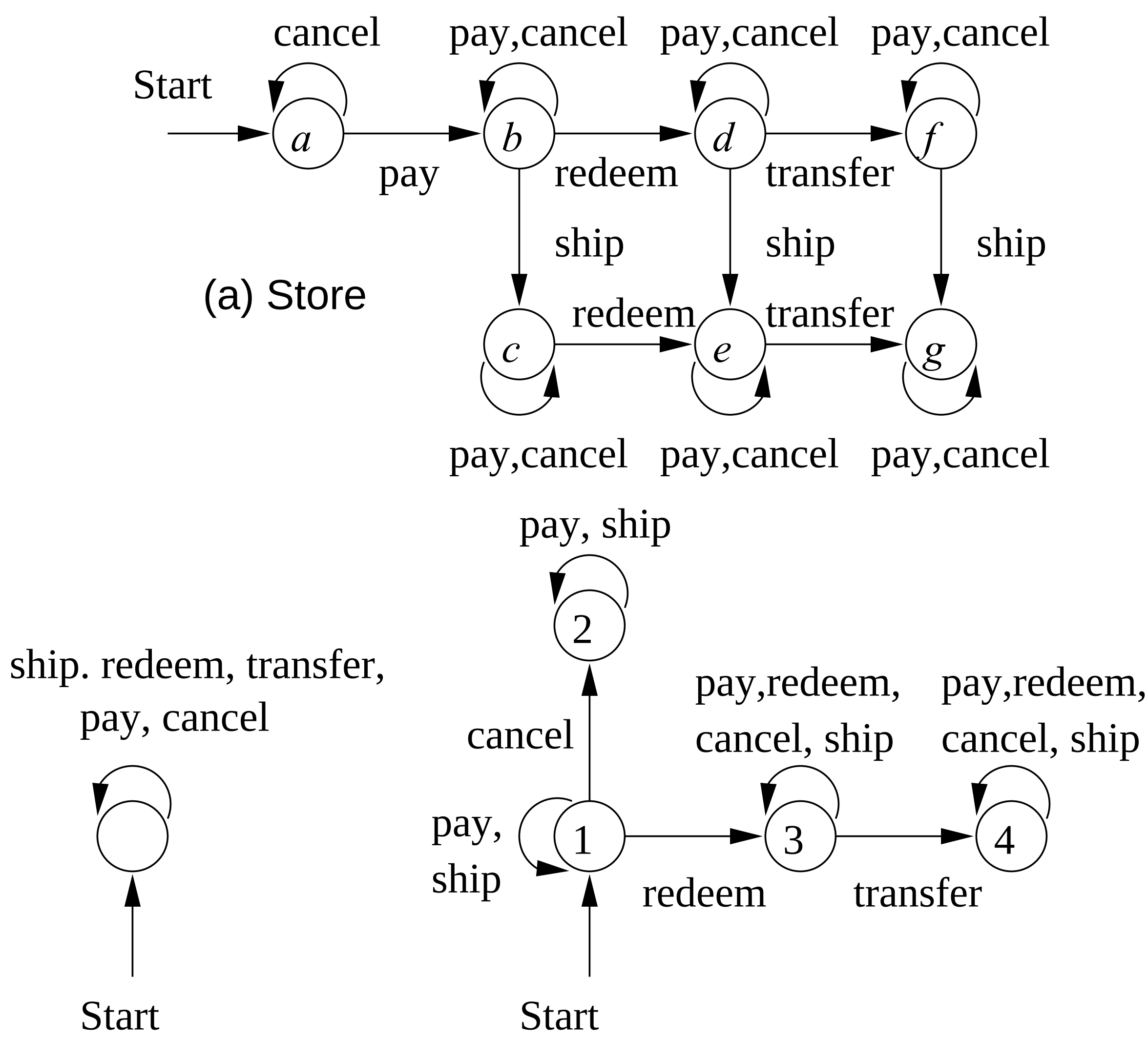


(b) Customer



(c) Bank

Completed protocols:

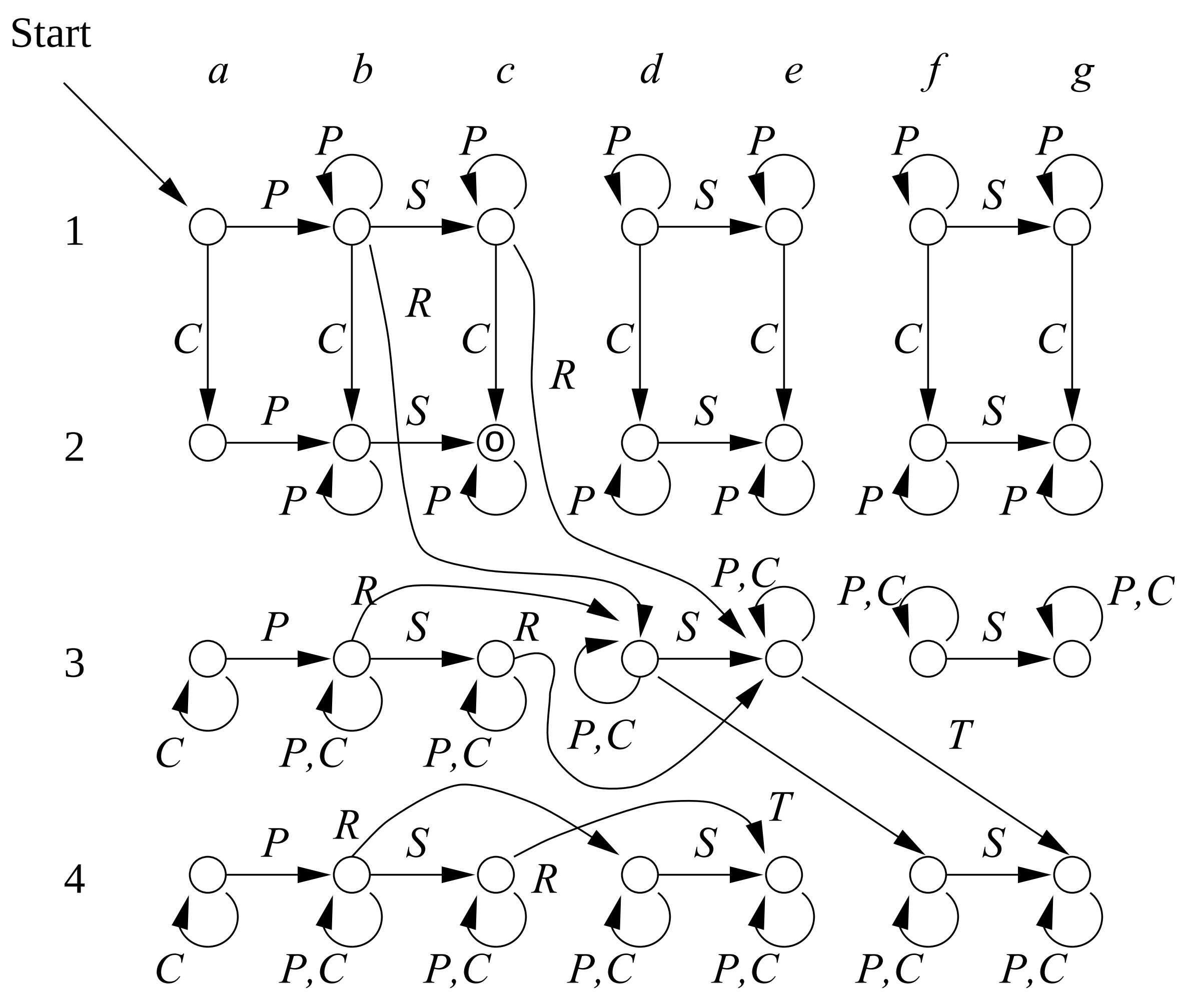


(a) Store

(b) Customer

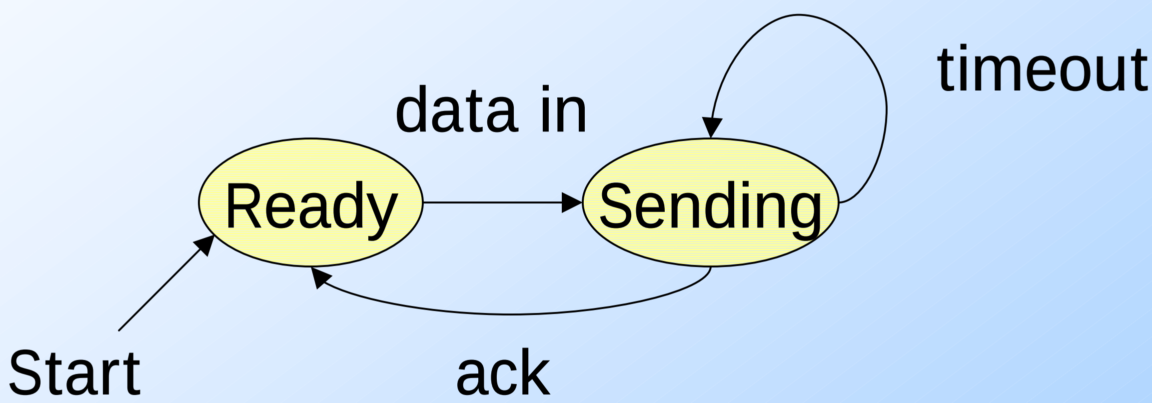
(c) Bank

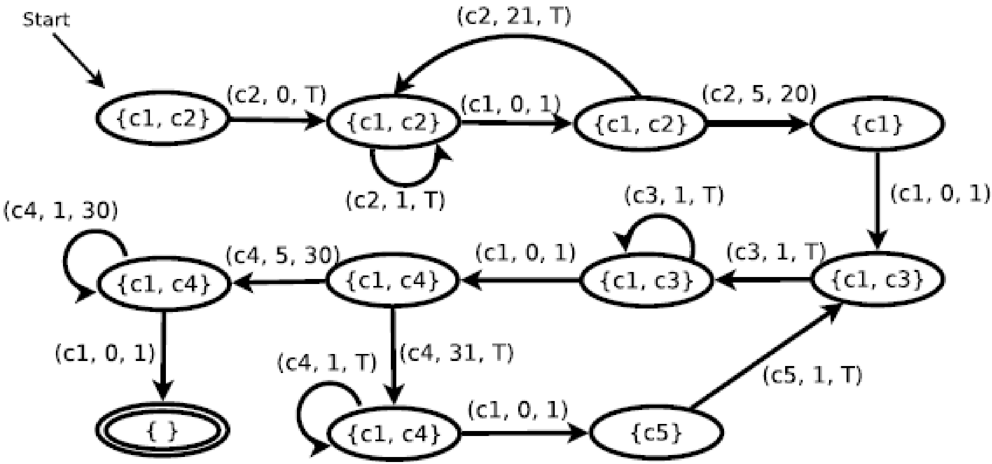
The entire system as an Automaton
(using Cartesian product):



More applications of FA can be found in Linz, Ch. 1.3.

Example: Protocol for Sending Data





c1. gettimeofday@plt	php5-fpm [0x42ee40 - 0x42ee7f]
c2. lcg_seed	php5-fpm [0x5eab00 - 0x5eab3f]
c3. php_gettimeofday	php5-fpm [0x5f0380 - 0x5f03bf]
c4. uniqid	php5-fpm [0x6028c0 - 0x6028ff]
c5. php_combined_lcg	php5-fpm [0x5eab40 - 0x5eab7f]

Figure 5: Attack NFA for case study in Sec. 6. Initial state q_0 indicated by "Start" and accepting states indicated with double ovals. T is the maximum Flush-Reload cycles without transitioning before the NFA stops accepting new inputs.

Computer Science > Machine Learning

[Submitted on 12 Oct 2018]

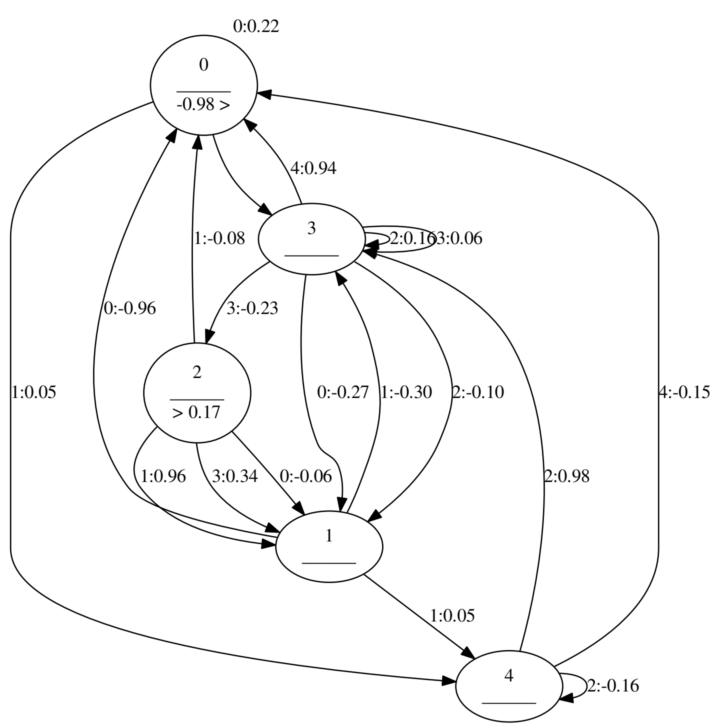
Explaining Black Boxes on Sequential Data using Weighted Automata

Stephane Ayache, Remi Eyraud, Noe Goudian

Understanding how a learned black box works is of crucial interest for the future of Machine Learning. In this paper, we pioneer the question of the global interpretability of learned black box models that assign numerical values to symbolic sequential data. To tackle that task, we propose a spectral algorithm for the extraction of weighted automata (WA) from such black boxes. This algorithm does not require the access to a dataset or to the inner representation of the black box: the inferred model can be obtained solely by querying the black box, feeding it with inputs and analyzing its outputs. Experiments using Recurrent Neural Networks (RNN) trained on a wide collection of 48 synthetic datasets and 2 real datasets show that the obtained approximation is of great quality.

Example of an extracted WA

Figure 5 gives the graphical representation on a WA extracted from a RNN trained on PAutomaC problem 24. This is not the best obtained WA on that dataset, but the metrics show that it is still a good approximation of the RNN.



2016 IEEE 36th International Conference on Distributed Computing Systems Workshops

What's so Different about Blockchain? – Blockchain is a Probabilistic State Machine –

Kenji Saito and Hiroyuki Yamada
Orb, Inc.
Sumitomo Bldg. 25F, 2-6-1 Nishi-Shinjuku, Shinjuku, Tokyo 163-0225, Japan
Email: {kenji | hiro}@imagine-orb.com[1]

A. *The Problem*

We are to understand the essential properties of blockchain that make it different from existing technology for reaching consensus and managing ledgers, such as Paxos[11] or its *byzantized* versions[4][6][12][13].

B. *Blockchain Consensus in Context of Consensus Problem*

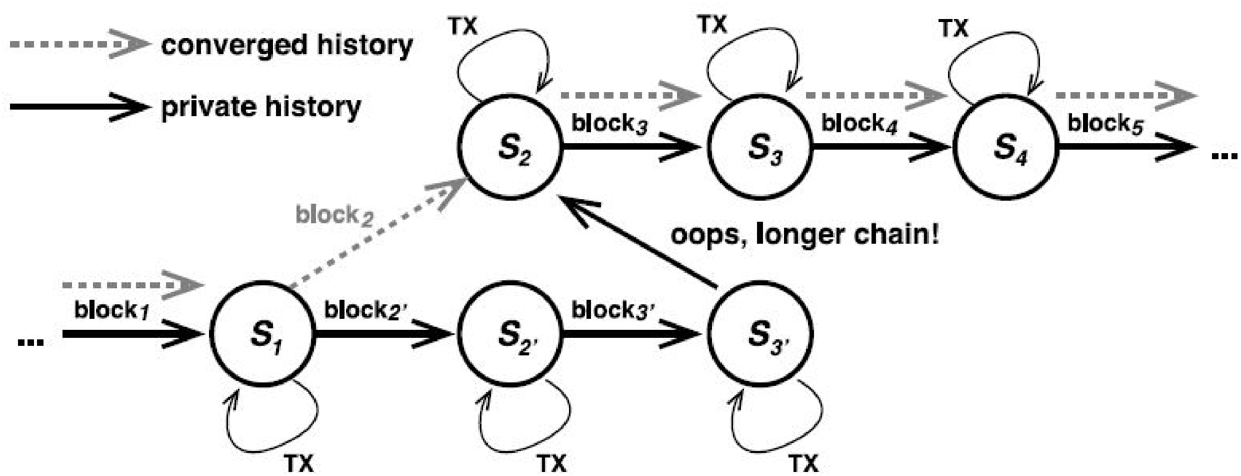
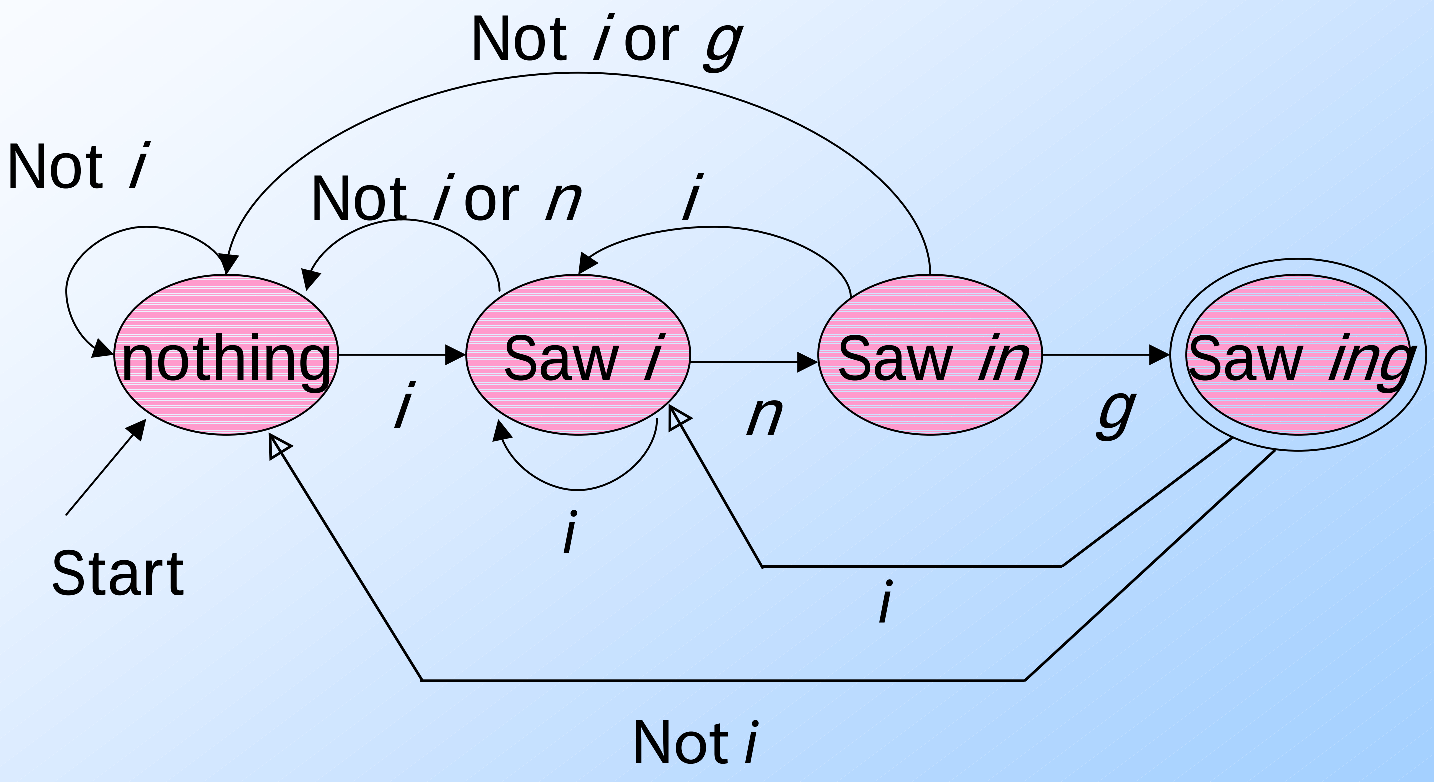


Fig. 4. Probabilistic State Machine of a Blockchain

Example: Recognizing Strings Ending in "ing"



Automata to Code

- In C/C++, make a piece of code for each state. This code:
  1. Reads the next input.
  2. Decides on the next state.
  3. Jumps to the beginning of the code for that state.

Example: Automata to Code

```
2: /* i seen */
  c = getNextInput();
  if (c == ’n’) goto 3;
  else if (c == ’i’) goto 2;
  else goto 1;
3: /* ”in” seen */
  . . .
```

Deterministic Finite Automata

A DFA is a quintuple

$$A = (Q, \Sigma, \delta, q_0, F)$$

- Q is a finite set of *states*
- Σ is a *finite alphabet* (=input symbols)
- δ is a *transition function* $(q, a) \mapsto p$ i.e., δ(q,a)=p
- $q_0 \in Q$ is the *start state*
- $F \subseteq Q$ is a set of *final states*

Example: An automaton A that accepts

$L = \{x01y : x, y \in \{0,1\}^*\}$

The automaton $A = (\{q_0, q_1, q_2\}, \{0,1\}, \delta, q_0, \{q_1\})$ as a *transition table*:

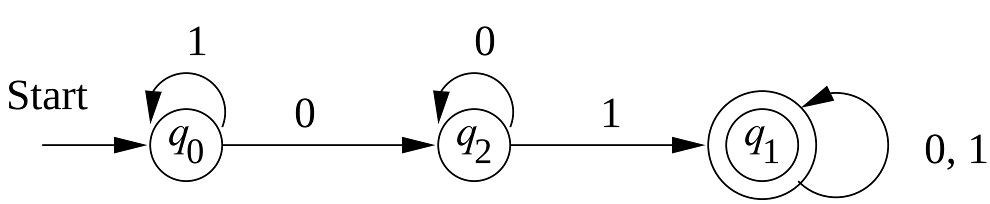
	0	1
$\rightarrow q_0$	q_2	q_0
$\star q_1$	q_1	q_1
q_2	q_2	q_1

$\hat{\delta}(q_0, 00) = q_2$

$\hat{\delta}(q_0, 01) = q_1$

$\hat{\delta}(q_2, 011) = q_1$

The automaton as a *transition diagram*:



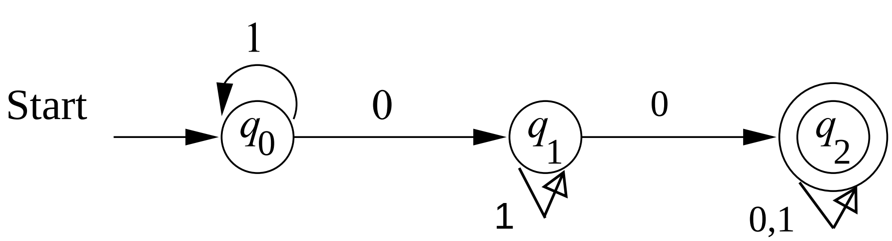
An FA *accepts* a string $w = a_1 a_2 \cdots a_n$ if there is a path in the transition diagram that

1. Begins at a start state

2. Ends at a final state
   or accepting

3. Has sequence of labels $a_1 a_2 \cdots a_n$ on the edges

Example: The FA



accepts e.g. the string 01101 and 1010, but not 110 or 0111

$\hat{\delta}(q,w)$: The state of the DFA after starting from state q and reading string w.

- The transition function δ can be extended to $\widehat{\delta}$ that operates on states and strings (as opposed to states and symbols)

Basis: $\widehat{\delta}(q, \epsilon) = q$

Induction: $\widehat{\delta}(q, xa) = \delta(\widehat{\delta}(q, x), a)$ for string x and symbol a

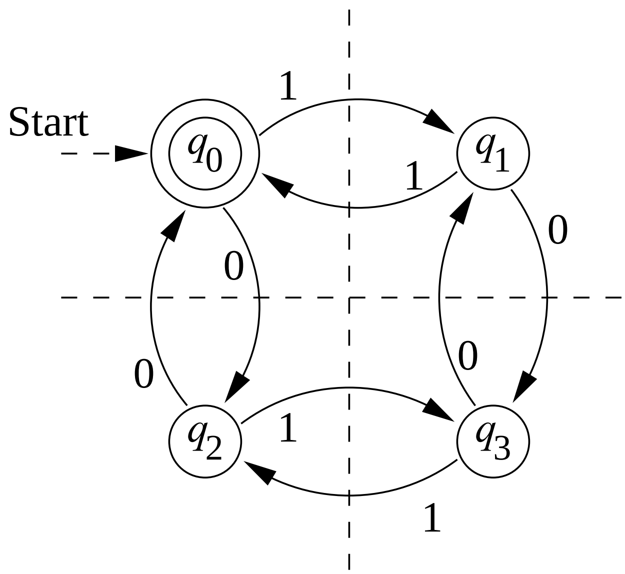
- Now, fomally, the *language accepted by* A is

$$L(A) = \{w : \widehat{\delta}(q_0, w) \in F\}$$

no more!
no less!

- The languages accepted by FA s are called *regular languages*

Example: DFA accepting all and only strings with an even number of 0's and an even number of 1's

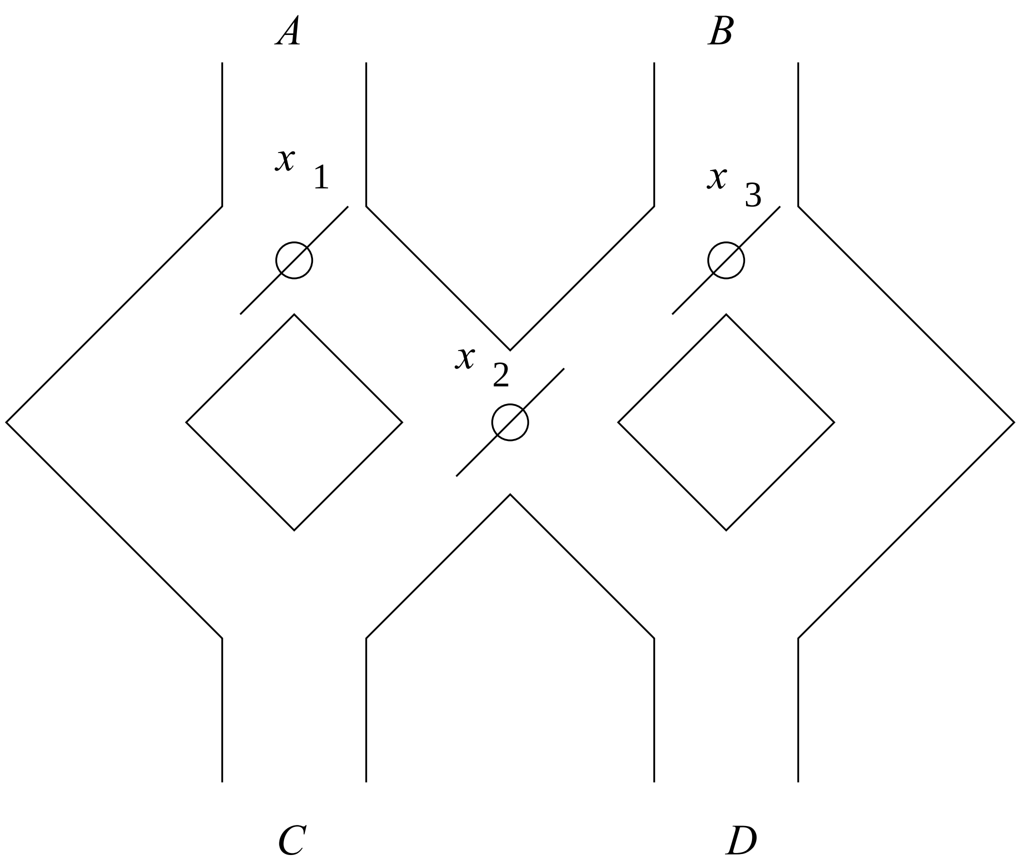


Tabular representation of the Automaton

	0	1
$\star \longrightarrow q_0$	q_2	q_1
q_1	q_3	q_0
q_2	q_0	q_3
q_3	q_1	q_2

Example

Marble-rolling toy from p. 53 of textbook



Ex. L_0 = {binary numbers divisible by 2}
L_1 = {binary numbers divisible by 3}
L_2 = {x | x $\in$ {0,1}*, x does not contain 000 as a substring}

A state is represented as sequence of three bits followed by r or a (previous input *rejected* or *accepted*)

For instance, $010a$, means
left, right, left, accepted

Tabular representation of DFA for the toy

	A	B
$\rightarrow 000r$	$100r$	$011r$
$\star 000a$	$100r$	$011r$
$\star 001a$	$101r$	$000a$
$010r$	$110r$	$001a$
$\star 010a$	$110r$	$001a$
$011r$	$111r$	$010a$
$100r$	$010r$	$111r$
$\star 100a$	$010r$	$111r$
$101r$	$011r$	$100a$
$\star 101a$	$011r$	$100a$
$110r$	$000a$	$101a$
$\star 110a$	$000a$	$101a$
$111r$	$001a$	$110a$

A View of the Parallel Computing Landscape. Par Lab, UC Berkeley. Communications of the ACM, 2009.

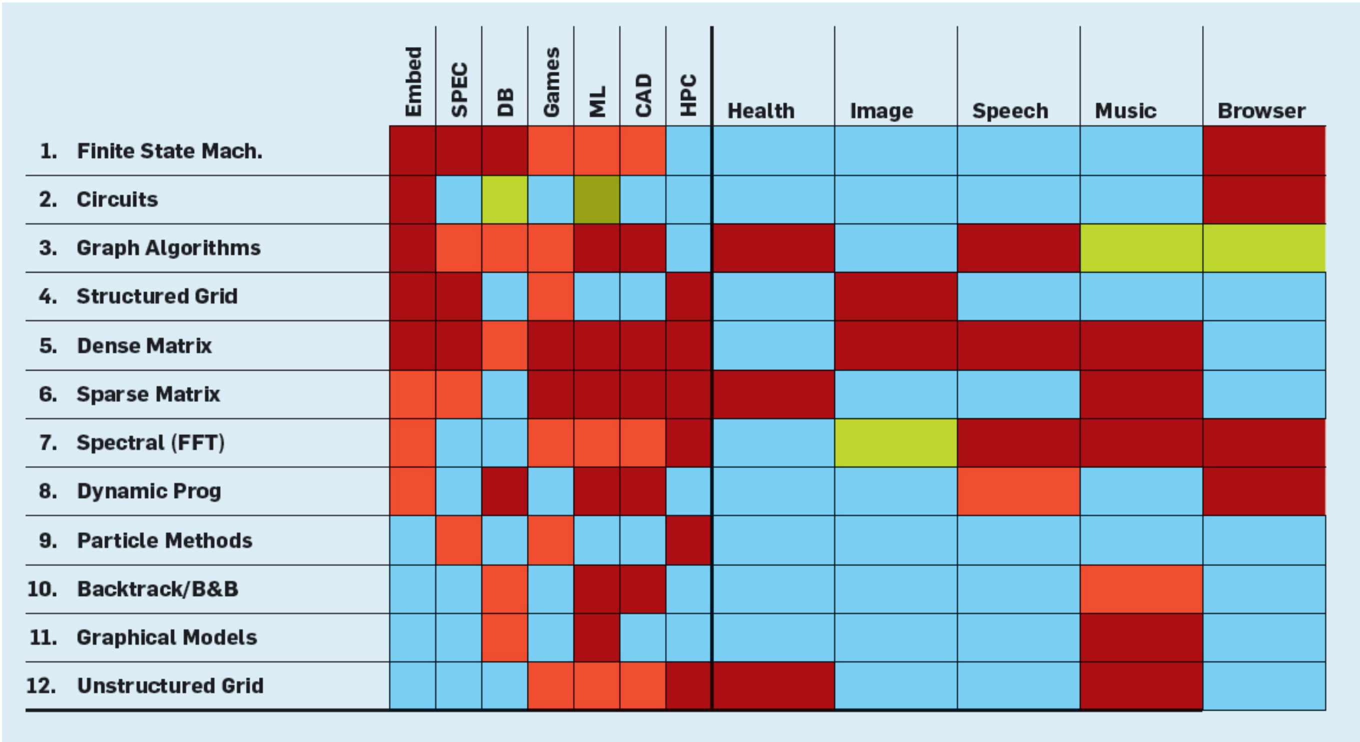
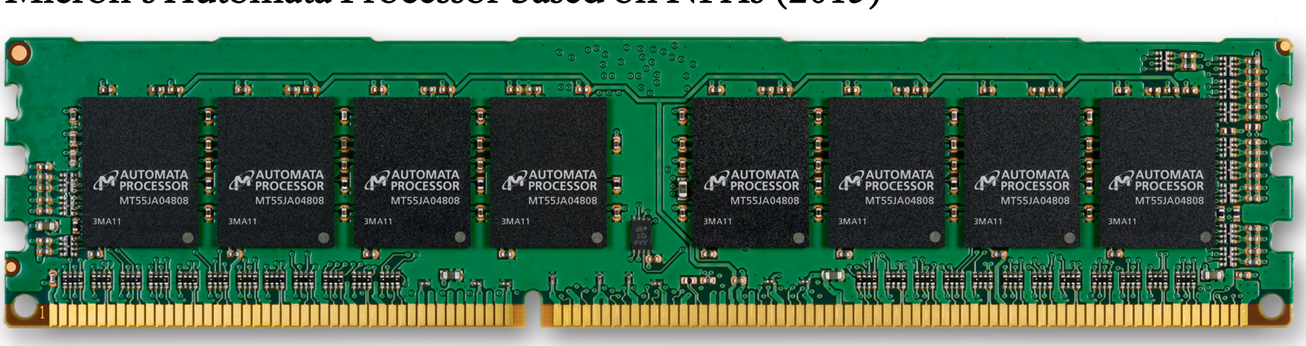
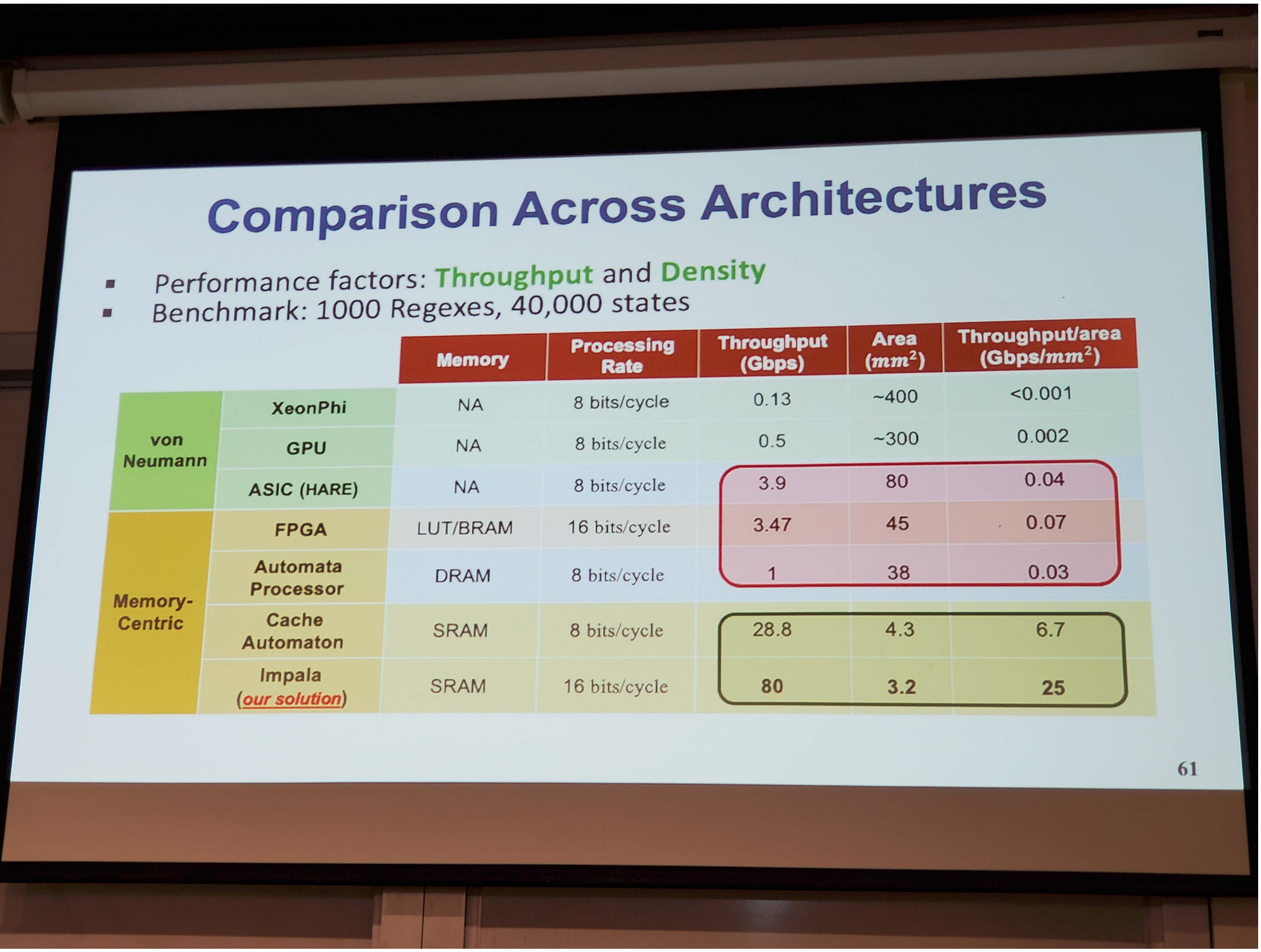


Figure 3. The color of a cell (for 12 computational patterns in several general application areas and five Par Lab applications) indicates the presence of that computational pattern in that application; red/high; orange/moderate; green/low; blue/rare.

Micron's Automata Processor based on NFAs (2013)



The Automata Processor (AP) is a completely new architecture for regular expression acceleration, including analysis, statistics, and logic operations. It scales to tens of thousands, even millions of processing elements for the largest challenges, with energy efficiency far greater than traditional CPUs and GPUs. It is much easier to program than FPGAs.


Comparison Across Architectures
Performance factors: Throughput and Density
Benchmark: 1000 Regexes, 40,000 states

		Memory	Processing Rate	Throughput (Gbps)	Area (mm^2)	Throughput/area (Gbps/mm^2)
von Neumann	XeonPhi	NA	8 bits/cycle	0.13	~400	<0.001
	GPU	NA	8 bits/cycle	0.5	~300	0.002
	ASIC (HARE)	NA	8 bits/cycle	3.9	80	0.04
Memory-Centric	FPGA	LUT/BRAM	16 bits/cycle	3.47	45	0.07
	Automata Processor	DRAM	8 bits/cycle	1	38	0.03
	Cache Automaton	SRAM	8 bits/cycle	28.8	4.3	6.7
	Impala (our solution)	SRAM	16 bits/cycle	**80**	**3.2**	**25**

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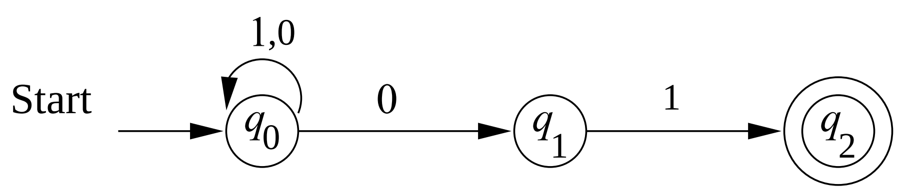
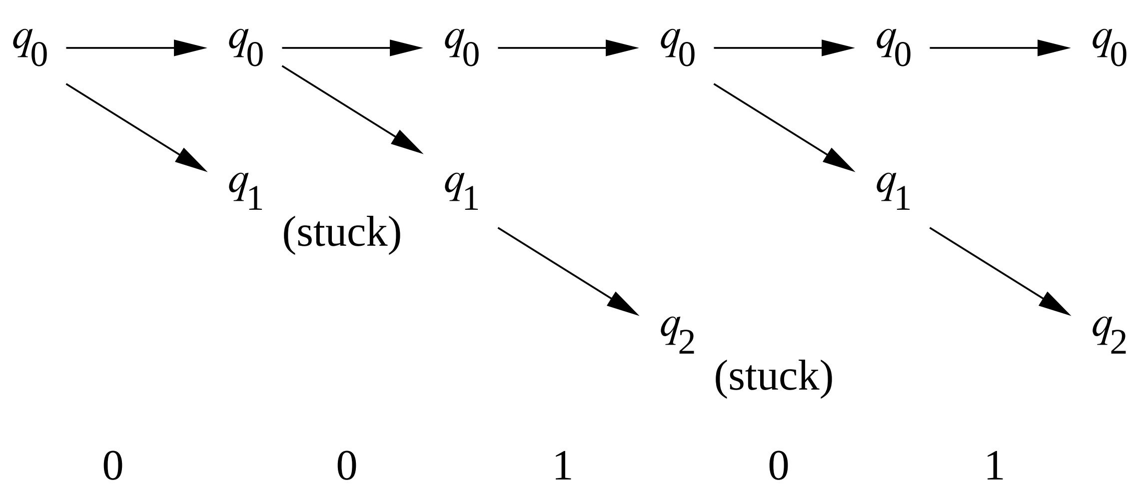
Nondeterministic Finite Automata

An NFA can be in several states at once, or, viewed another way, it can "guess" which state to go to next

Example: An automaton that accepts all and only strings ending in 01.



Here is what happens when the NFA processes the input 00101



Formally, an NFA is a quintuple

$$A = (Q, \Sigma, \delta, q_0, F)$$

- Q is a finite set of states
- Σ is a finite alphabet
- δ is a transition function from $Q \times \Sigma$ to the *powerset* of Q
- $q_0 \in Q$ is the *start state*
- $F \subseteq Q$ is a set of *final states*

Example: The NFA from the previous slide is

$$(\{q_0, q_1, q_2\}, \{0, 1\}, \delta, q_0, \{q_2\})$$

where δ is the transition function

	0	1
$\rightarrow q_0$	$\{q_0, q_1\}$	$\{q_0\}$
q_1	$\emptyset$	$\{q_2\}$
$\star q_2$	$\emptyset$	$\emptyset$

Extended transition function $\widehat{\delta}$.

Basis: $\widehat{\delta}(q, \epsilon) = \{q\}$

Induction:

$$\widehat{\delta}(q, xa) = \bigcup_{p \in \widehat{\delta}(q,x)} \delta(p, a)$$

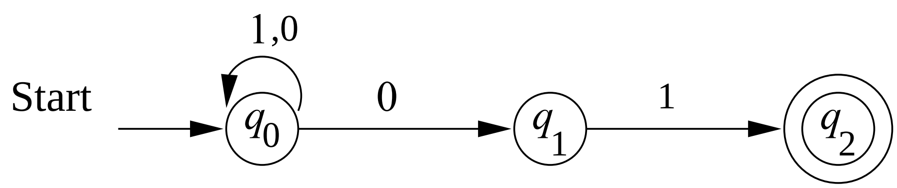
where x is a string
and a is a symbol

Example: Let's compute $\widehat{\delta}(q_0, 00101)$ on the blackboard. How about $\hat{\delta}(q_0,0010)$?

- Now, fomally, the *language accepted by A* is

$$L(A) = \{w : \widehat{\delta}(q_0, w) \cap F \neq \emptyset\}$$

Let's prove formally that the NFA



accepts the language $\{x01 : x \in \Sigma^*\}$. We'll do a mutual induction for the three statements below based on $|w|$:

0. $w \in \Sigma^* \Rightarrow q_0 \in \widehat{\delta}(q_0, w)$

1. $q_1 \in \widehat{\delta}(q_0, w) \Leftrightarrow w = x0$

2. $q_2 \in \widehat{\delta}(q_0, w) \Leftrightarrow w = x01$

Basis: If $|w| = 0$ then $w = \epsilon$. Then statement (0) follows from def. For (1) and (2) both sides are false for ϵ

Induction: Assume $w = xa$, where $a \in \{0, 1\}$, $|x| = n$ and statements (0)–(2) hold for x. We will show on the blackboard in class that the statements hold for xa.

Ex 1. Design an NFA for
  L = {x | x $\in$ {0,1}*, the 3rd last bit of x is a 1}
How many states would be required in the DFA for L?

Ex 2. Design an NFA for the language that contains binary strings with either two consecutive 0's or two consecutive 1's.

Equivalence of DFA and NFA

- NFA's are usually easier to "program" in.

- Surprisingly, for any NFA N there is a DFA D, such that $L(D) = L(N)$, and vice versa.

- This involves the *subset construction*, an important example how an automaton B can be generically constructed from another automaton A.

- Given an NFA

$$N = (Q_N, \Sigma, \delta_N, q_0, F_N)$$

we will construct a DFA

$$D = (Q_D, \Sigma, \delta_D, \{q_0\}, F_D)$$

such that

$$L(D) = L(N)$$

.

The details of the subset construction:

- $Q_D = \{S : S \subseteq Q_N\}$.

Note: $|Q_D| = 2^{|Q_N|}$, although most states in Q_D are likely to be garbage.

- $F_D = \{S \subseteq Q_N : S \cap F_N \neq \emptyset\}$

- For every $S \subseteq Q_N$ and $a \in \Sigma$,

$$\delta_D(S, a) = \bigcup_{p \in S} \delta_N(p, a)$$

Let's construct δ_D from the NFA on slide 27

	0	1
$\emptyset$	$\emptyset$	$\emptyset$
$\rightarrow \{q_0\}$	$\{q_0, q_1\}$	$\{q_0\}$
$\{q_1\}$	$\emptyset$	$\{q_2\}$
$\star\{q_2\}$	$\emptyset$	$\emptyset$
$\{q_0, q_1\}$	$\{q_0, q_1\}$	$\{q_0, q_2\}$
$\star\{q_0, q_2\}$	$\{q_0, q_1\}$	$\{q_0\}$
$\star\{q_1, q_2\}$	$\emptyset$	$\{q_2\}$
$\star\{q_0, q_1, q_2\}$	$\{q_0, q_1\}$	$\{q_0, q_2\}$

Note: The states of D correspond to subsets of states of N, but we could have denoted the states of D by, say, $A - F$ just as well.

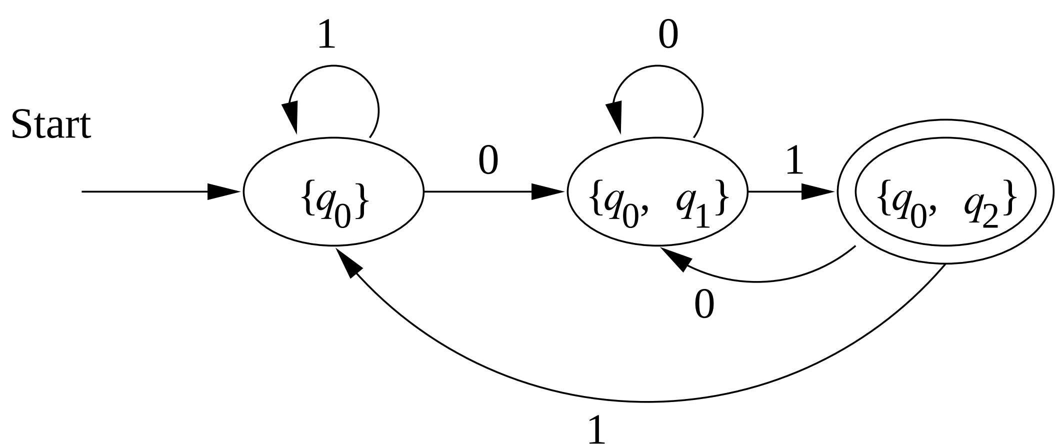
	0	1
A	A	A
$\rightarrow B$	E	B
C	A	D
$\star D$	A	A
E	E	F
$\star F$	E	B
$\star G$	A	D
$\star H$	E	F

We can often avoid the exponential blow-up by constructing the transition table for D only for accessible states S as follows:

Basis: $S = \{q_0\}$ is accessible in D

Induction: If state S is accessible, so are the states in $\bigcup_{a \in \Sigma} \{\delta_D(S, a)\}$

Example: The "subset" DFA with accessible states only.



Theorem 2.11: Let D be the "subset" DFA of an NFA N. Then $L(D) = L(N)$.

Proof: First we show by an induction on $|w|$ that

$$\widehat{\delta}_D(\{q_0\}, w) = \widehat{\delta}_N(q_0, w)$$

Basis: $w = \epsilon$. The claim follows from def.

Induction:

$$
\begin{aligned}
\widehat{\delta}_D(\{q_0\}, xa) &\stackrel{\text{def}}{=} \delta_D(\widehat{\delta}_D(\{q_0\}, x), a) \\
&\stackrel{\text{i.h.}}{=} \delta_D(\widehat{\delta}_N(q_0, x), a) \\
&\stackrel{\text{cst}}{=} \bigcup_{p \in \widehat{\delta}_N(q_0, x)} \delta_N(p, a) \\
&\stackrel{\text{def}}{=} \widehat{\delta}_N(q_0, xa)
\end{aligned}
$$

Now (**why?**) it follows that $L(D) = L(N)$.

Theorem 2.12: A language L is accepted by some DFA if and only if L is accepted by some NFA.

Proof: The "if" part is Theorem 2.11.

For the "only if" part we note that any DFA can be converted to an equivalent NFA by modifying the δ_D to δ_N by the rule

- If $\delta_D(q,a) = p$, then $\delta_N(q,a) = \{p\}$.

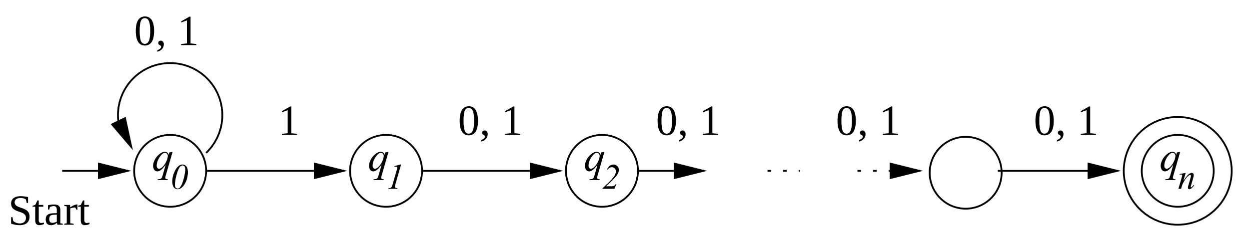
By induction on $|w|$ it will be shown in the tutorial that if $\hat{\delta}_D(q_0, w) = p$, then $\hat{\delta}_N(q_0, w) = \{p\}$.

The claim of the theorem follows.

How do you convert an NFA to C/C++ code?

Exponential Blow-Up

There is an NFA N with $n+1$ states that has no equivalent DFA with fewer than 2^n states



$$L(N) = \{x1c_2c_3\cdots c_n : x \in \{0,1\}^*, c_i \in \{0,1\}\}$$

Suppose an equivalent DFA D with fewer than 2^n states exists.

D must remember the last n symbols it has read, but how?

There are 2^n bitsequences $a_1a_2\cdots a_n$

$$\exists\, q, a_1a_2\cdots a_n, b_1b_2\cdots b_n : q = \hat{\delta}_D(q_0, a_1a_2\cdots a_n),$$
$$q = \hat{\delta}_D(q_0, b_1b_2\cdots b_n),$$
$$a_1a_2\cdots a_n \neq b_1b_2\cdots b_n$$

Case 1:

$1a_2 \cdots a_n$
$0b_2 \cdots b_n$

Then q has to be both an accepting and a nonaccepting state.

Case 2:

$a_1 \cdots a_{i-1}1a_{i+1} \cdots a_n$
$b_1 \cdots b_{i-1}0b_{i+1} \cdots b_n$

Now $\hat{\delta}_{\mathrm{D}}(q_0, a_1 \cdots a_{i-1}1a_{i+1} \cdots a_n0^{i-1}) =$
$\hat{\delta}_{\mathrm{D}}(q_0, b_1 \cdots b_{i-1}0b_{i+1} \cdots b_n0^{i-1})$

and $\hat{\delta}_{\mathrm{D}}(q_0, a_1 \cdots a_{i-1}1a_{i+1} \cdots a_n0^{i-1}) \in F_D$

$$\hat{\delta}_{\mathrm{D}}(q_0, b_1 \cdots b_{i-1}0b_{i+1} \cdots b_n0^{i-1}) \notin F_D$$

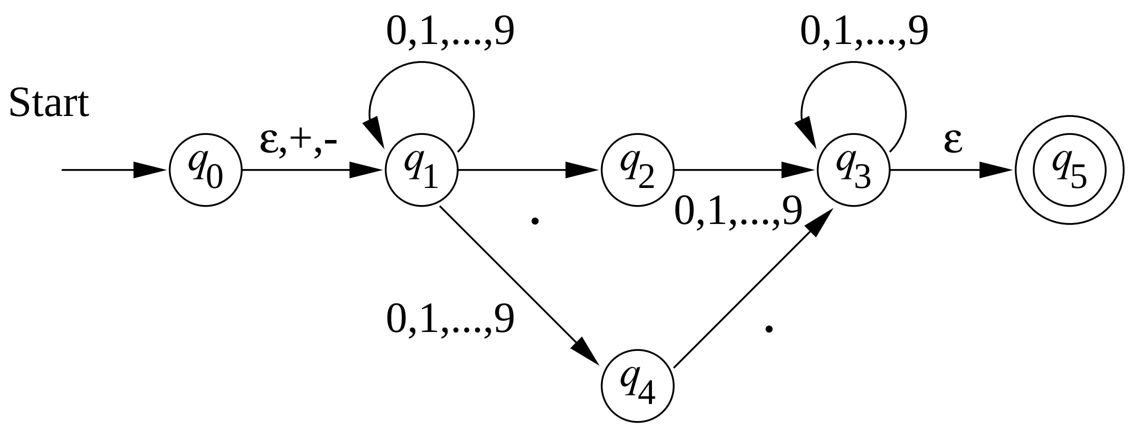
FA's with Epsilon-Transitions

An ϵ-NFA accepting decimal numbers consisting of:

1. An optional + or - sign
2. A string of digits
3. a decimal point
4. another string of digits

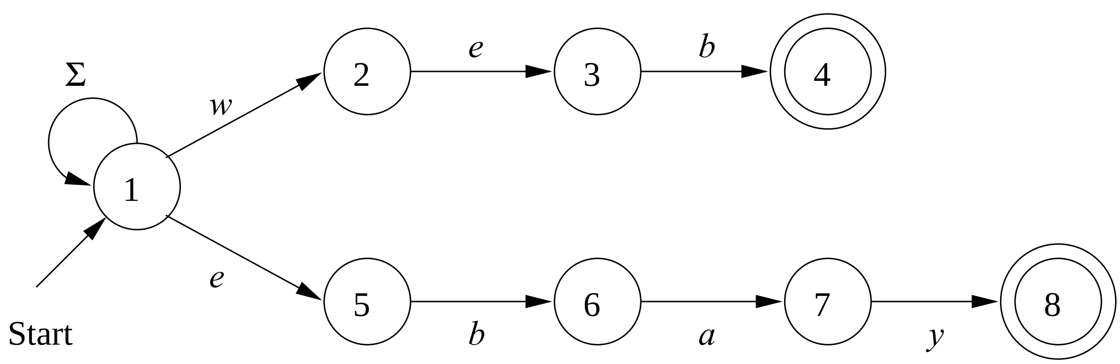
E.g. -12.5
+10.00
5.
-.6

One of the strings (2) and (4) is optional.



Example:

ϵ-NFA accepting the set of keywords $\{\texttt{ebay}, \texttt{web}\}$



Instead of this NFA, we can construct an ε–NFA that has an ε-move for each keyword.

Ex. Design an NFA for
L = {x | x ∈ {0,1}*, x begins or ends with 00}

An ϵ-NFA is a quintuple $(Q, \Sigma, \delta, q_0, F)$ where δ is a function from $Q \times (\Sigma \cup \{\epsilon\})$ to the powerset of Q.

Example: The ϵ-NFA from the previous slide

$$E = (\{q_0, q_1, \ldots, q_5\}, \{., +, -, 0, 1, \ldots, 9\}\ \delta, q_0, \{q_5\})$$

where the transition table for δ is

	ϵ	+,-	.	$0, \ldots, 9$
$\rightarrow q_0$	$\{q_1\}$	$\{q_1\}$	$\emptyset$	$\emptyset$
q_1	$\emptyset$	$\emptyset$	$\{q_2\}$	$\{q_1, q_4\}$
q_2	$\emptyset$	$\emptyset$	$\emptyset$	$\{q_3\}$
q_3	$\{q_5\}$	$\emptyset$	$\emptyset$	$\{q_3\}$
q_4	$\emptyset$	$\emptyset$	$\{q_3\}$	$\emptyset$
$\star q_5$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

$\boxed{\text{ECLOSE}}$ or ε-closure

We close a state by adding all states reachable by a sequence $\epsilon\epsilon \cdots \epsilon$

Inductive definition of $\text{ECLOSE}(q)$

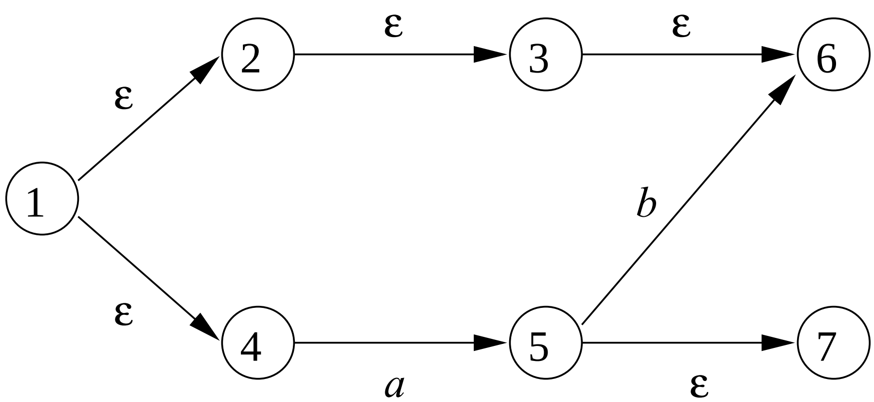
Basis:

$q \in \text{ECLOSE}(q)$

Induction:

$p \in \text{ECLOSE}(q)$ and $r \in \delta(p, \epsilon) \quad \Rightarrow$
$r \in \text{ECLOSE}(q)$

Example of ϵ-closure



For instance,

$$\text{ECLOSE}(1) = \{1, 2, 3, 4, 6\}$$

- Inductive definition of $\widehat{\delta}$ for ϵ-NFA's

Basis:

$$\widehat{\delta}(q, \epsilon) = \text{ECLOSE}(q)$$

Induction:

$$\widehat{\delta}(q, xa) = \bigcup_{p \in \hat{\delta}(q,x)} \text{ECLOSE}(\delta(p, a))$$

Let's compute on the blackboard in class $\widehat{\delta}(q_0, 5.6)$ for the NFA on slide 43

$\hat{\delta}(q_0,\varepsilon) = \text{ECLOSE}(q_0) = \{q_0,q_1\}$

$\hat{\delta}(q_0,5) = \text{ECLOSE}(\{q_1,q_4\}) = \{q_1,q_4\}$, because $\delta(q_0,5) \cup \delta(q_1,5) = \{q_1,q_4\}$

$\hat{\delta}(q_0,5.) = \text{ECLOSE}(\{q_2,q_3\}) = \{q_2,q_3,q_5\}$

$\hat{\delta}(q_0,5.6) = \text{ECLOSE}(\{q_3\}) = \{q_3,q_5\}$

Given an ϵ-NFA

$$E = (Q_E, \Sigma, \delta_E, q_0, F_E)$$

we will construct a DFA

$$D = (Q_D, \Sigma, \delta_D, q_D, F_D)$$

such that

$$L(D) = L(E)$$

Details of the construction:

- $Q_D = \{S : S \subseteq Q_E \text{ and } S = \text{ECLOSE}(S)\}$

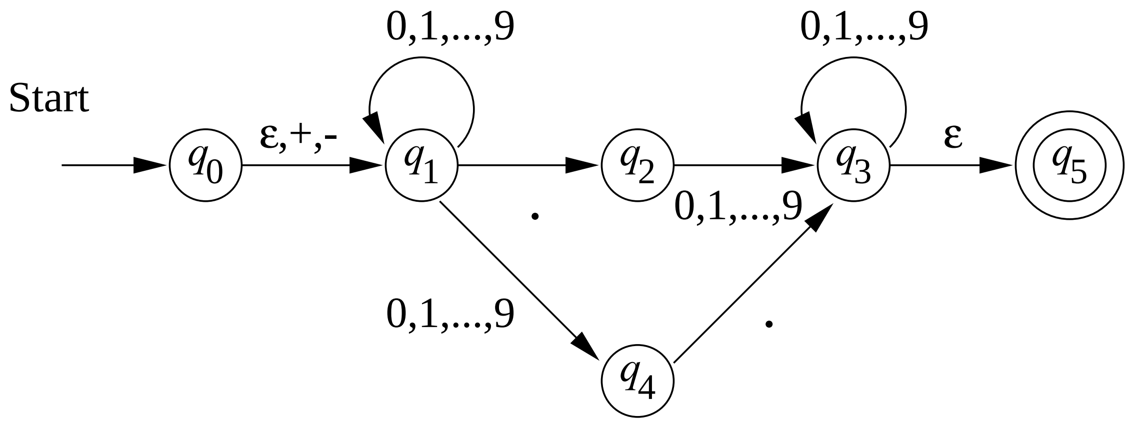
- $q_D = \text{ECLOSE}(q_0)$

- $F_D = \{S : S \in Q_D \text{ and } S \cap F_E \neq \emptyset\}$

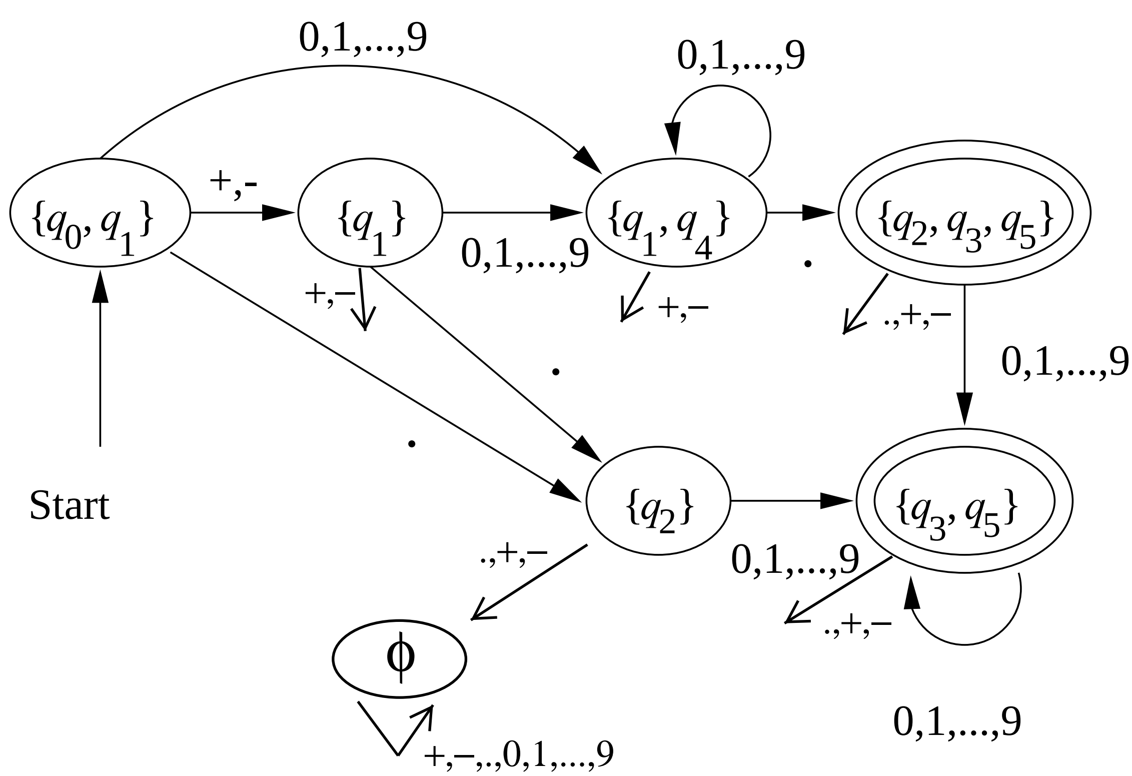
- $\delta_D(S, a) =$

  $\bigcup\{\text{ECLOSE}(p) : p \in \delta_E(t, a) \text{ for some } t \in S\}$

Example: ϵ-NFA E



DFA D corresponding to E



Theorem 2.22: A language L is accepted by some ϵ-NFA E if and only if L is accepted by some DFA.

Proof: We use D constructed as above and show by induction that $\widehat{\delta}_D(q_D, w) = \widehat{\delta}_E(q_0, w)$

Basis: $\widehat{\delta}_E(q_0, \epsilon) = \text{ECLOSE}(q_0) = q_D = \widehat{\delta}(q_D, \epsilon)$

Induction:

$$\hat{\delta}_E(q_0, xa) \stackrel{\text{DEF}}{=} \bigcup_{p \in \hat{\delta}_E(q_0, x)} \text{ECLOSE}(\delta_E(p, a))$$

$$\stackrel{\text{I.H.}}{=} \bigcup_{p \in \hat{\delta}_D(q_D, x)} \text{ECLOSE}(\delta_E(p, a))$$

$$\stackrel{\text{CST}}{=} \delta_D(\hat{\delta}_D(q_D, x), a)$$

$$\stackrel{\text{DEF}}{=} \hat{\delta}_D(q_D, xa)$$

Regular expressions

An FA (NFA or DFA) is a "blueprint" for contructing a machine recognizing a regular language.

A *regular expression* is a "user-friendly," declarative way of describing a regular language.

Example: $\mathbf{01}^* + \mathbf{10}^*$

Regular expressions are used in e.g.

1. UNIX `grep` command

   grep PATTERN FILE

2. UNIX Lex (Lexical analyzer generator) and Flex (Fast Lex) tools.

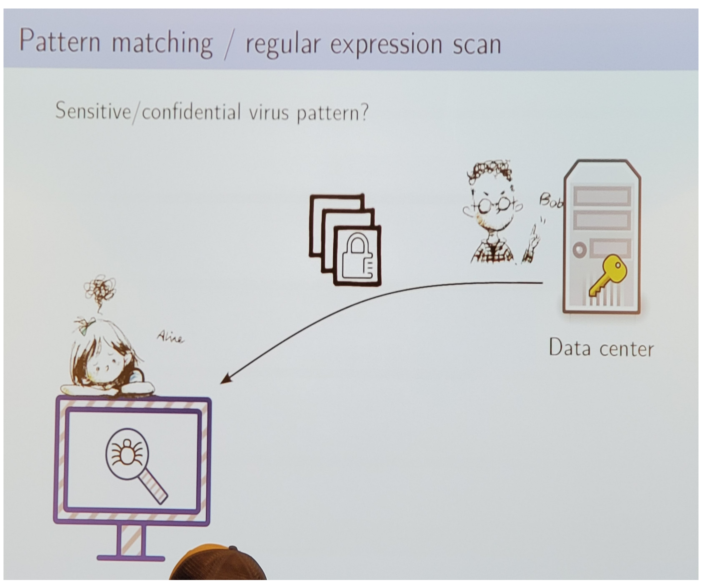
3. Text/email mining (e.g., for HomeUnion, one of the two languages for Micron's Automata Processor)

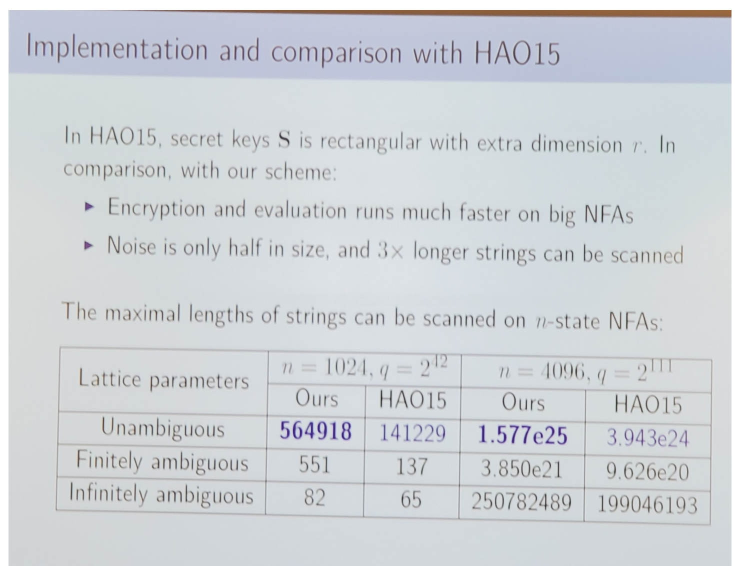
Homomorphic Encryption for Finite Automata

Authors: Nicholas Genise, Craig Gentry, Shai Halevi, Baiyu Li, Daniele Micciancio

Publisher: Springer International Publishing

Published in: Advances in Cryptology – ASIACRYPT 2019





Implementation and comparison with HAO15

In HAO15, secret keys $\mathbf{S}$ is rectangular with extra dimension r. In comparison, with our scheme:

- Encryption and evaluation runs much faster on big NFAs
- Noise is only half in size, and $3\times$ longer strings can be scanned

The maximal lengths of strings can be scanned on n-state NFAs:

Lattice parameters	$n = 1024, q = 2^{42}$		$n = 4096, q = 2^{111}$	
	Ours	HAO15	Ours	HAO15
Unambiguous	**564918**	141229	**1.577e25**	3.943e24
Finitely ambiguous	551	137	3.850e21	9.626e20
Infinitely ambiguous	82	65	250782489	199046193

Operations on languages

Union:

$L \cup M = \{w : w \in L$ or $w \in M\}$

Concatenation:

$L \cdot M = \{w : w = xy, x \in L, y \in M\}$

E.g., $\{0^2, 0^4\} \cdot \{1, 1^3, 1^5\}$
$= \{0^2 1, 0^2 1^3, 0^2 1^5, 0^4 1, 0^4 1^3, 0^4 1^5\}$

Powers:

$L^0 = \{\epsilon\}, L^1 = L, L^{k+1} = L \cdot L^k$

Kleene Closure:

$$L^* = \bigcup_{i=0}^{\infty} L^i$$

Question: What are $\emptyset^0$, $\emptyset^i$, and $\emptyset^*$

Question: What is $\{0^2, 0^3\}^*$?

Building regex's

Inductive definition of regex's:

Basis: $\boldsymbol{\epsilon}$ is a regex and $\boldsymbol{\emptyset}$ is a regex.
$L(\boldsymbol{\epsilon}) = \{\epsilon\}$, and $L(\boldsymbol{\emptyset}) = \emptyset$.

If $a \in \Sigma$, then $\boldsymbol{a}$ is a regex.
$L(\boldsymbol{a}) = \{a\}$.

Induction:

If E is a regex's, then (E) is a regex.
$L((E)) = L(E)$.

If E and F are regex's, then $E + F$ is a regex.
$L(E + F) = L(E) \cup L(F)$.

If E and F are regex's, then $E{\cdot}F$ (or simply EF) is a regex. $L(E{\cdot}F) = L(E){\cdot}L(F)$.

If E is a regex's, then $E^\star$ is a regex.
$L(E^\star) = (L(E))^*$.

Example: Regex for

$L = \{w \in \{0,1\}^* :$ 0 and 1 alternate in $w\}$

$$(\mathbf{01})^* + (\mathbf{10})^* + \mathbf{0}(\mathbf{10})^* + \mathbf{1}(\mathbf{01})^*$$

or, equivalently,

$$(\boldsymbol{\epsilon} + \mathbf{1})(\mathbf{01})^*(\boldsymbol{\epsilon} + \mathbf{0})$$

Order of precedence for operators:

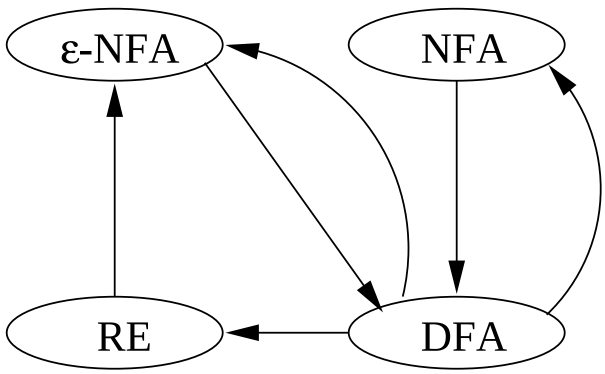
1. Star
2. Dot
3. Plus

Example: $\mathbf{01}^* + \mathbf{1}$ is grouped $(\mathbf{0}(\mathbf{1}^*)) + \mathbf{1}$

Ex. Regex's for $L_1 = \{ w \mid w \in \{0,1\}^*, w$ contains no consecutive 0's$\}$
$L_2 = \{ w \mid w \in \{0,1\}^*,$ the number of 0's in w is even$\}$.

Equivalence of FA's and regex's

We have already shown that DFA's, NFA's, and ϵ-NFA's all are equivalent.



To show FA's equivalent to regex's we need to establish that

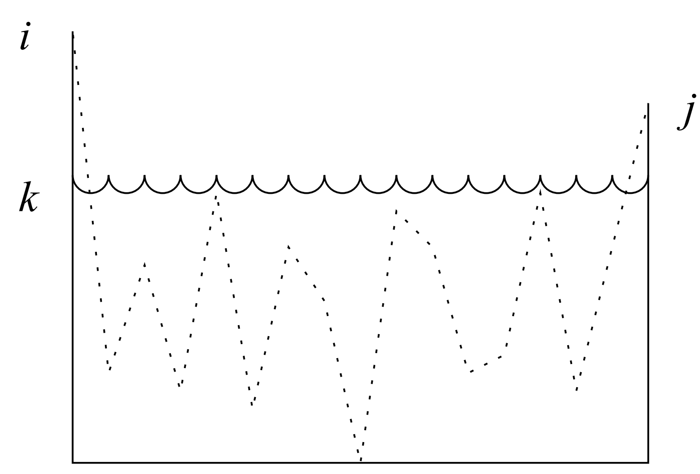
1. For every DFA A we can find (construct, in this case) a regex R, s.t. $L(R) = L(A)$.

2. For every regex R there is an ϵ-NFA A, s.t. $L(A) = L(R)$.

Theorem 3.4: For every DFA $A = (Q, \Sigma, \delta, q_0, F)$ there is a regex R, s.t. $L(R) = L(A)$.

Proof: Let the states of A be $\{1, 2, \ldots, n\}$, with 1 being the start state.

- Let $R_{ij}^{(k)}$ be a regex describing the set of labels of all paths in A from state i to state j going through intermediate states $\{1, \ldots, k\}$ only. Note that, i and j don't have to be in {1, ...,k}.



$R_{ij}^{(k)}$ will be defined inductively. Note that

$$L\left(\bigoplus_{j\in F} {R_{1j}}^{(n)}\right) = L(A)$$

Basis: $k = 0$, i.e. no intermediate states.

- *Case 1:* $i \neq j$ $\boxed{\text{i.e., arc i -> j}}$

$$R_{ij}^{(0)} = \bigoplus_{\{a\in\Sigma:\delta(i,a)=j\}} \boldsymbol{a}$$

- *Case 2:* $i = j$ $\boxed{\text{i.e., arc i -> i or } \varepsilon}$

$$R_{ii}^{(0)} = \left(\bigoplus_{\{a\in\Sigma:\delta(i,a)=i\}} \boldsymbol{a}\right) + \boldsymbol{\epsilon}$$

Induction:

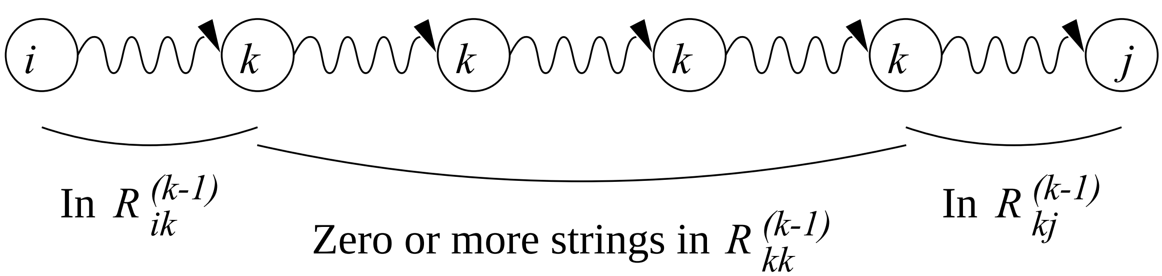
$$R_{ij}^{(k)}$$

$$=$$

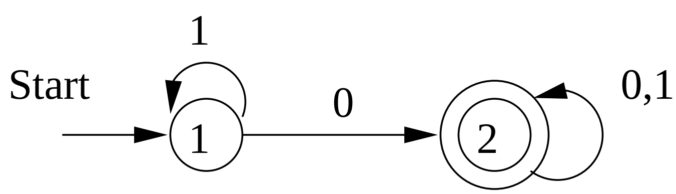
$$R_{ij}^{(k-1)} \qquad \boxed{\text{does not go through k}}$$

$$+$$

$$R_{ik}^{(k-1)}\left(R_{kk}^{(k-1)}\right)^{*}R_{kj}^{(k-1)} \qquad \boxed{\begin{array}{l}\text{goes through k}\\ \text{at least once}\end{array}}$$



Example: Let's find R for A, where
$L(A) = \{x0y : x \in \{1\}^* \text{ and } y \in \{0,1\}^*\}$



$R_{11}^{(0)}$	$\boldsymbol{\epsilon + 1}$
$R_{12}^{(0)}$	$\mathbf{0}$
$R_{21}^{(0)}$	$\emptyset$
$R_{22}^{(0)}$	$\boldsymbol{\epsilon + 0 + 1}$

We will need the following *simplification rules:*

- $(\epsilon + R)^* = R^*$ (ε+R)R* = R*
- $R + RS^* = RS^*$ ε+R+R* = R*
- $\emptyset R = R\emptyset = \emptyset$ (Annihilation)
- $\emptyset + R = R + \emptyset = R$ (Identity)

$R_{11}^{(0)}$	$\boldsymbol{\epsilon + 1}$
$R_{12}^{(0)}$	$\mathbf{0}$
$R_{21}^{(0)}$	$\emptyset$
$R_{22}^{(0)}$	$\boldsymbol{\epsilon + 0 + 1}$

$$R_{ij}^{(1)} = R_{ij}^{(0)} + R_{i1}^{(0)}\left(R_{11}^{(0)}\right)^* R_{1j}^{(0)}$$

	By direct substitution	Simplified
$R_{11}^{(1)}$	$\boldsymbol{\epsilon + 1 + (\epsilon + 1)(\epsilon + 1)^*(\epsilon + 1)}$	$\mathbf{1^*}$
$R_{12}^{(1)}$	$\boldsymbol{0 + (\epsilon + 1)(\epsilon + 1)^*0}$	$\mathbf{1^*0}$
$R_{21}^{(1)}$	$\boldsymbol{\emptyset + \emptyset(\epsilon + 1)^*(\epsilon + 1)}$	$\emptyset$
$R_{22}^{(1)}$	$\boldsymbol{\epsilon + 0 + 1 + \emptyset(\epsilon + 1)^*0}$	$\boldsymbol{\epsilon + 0 + 1}$

	Simplified
$R_{11}^{(1)}$	$\mathbf{1}^*$
$R_{12}^{(1)}$	$\mathbf{1}^*\mathbf{0}$
$R_{21}^{(1)}$	$\emptyset$
$R_{22}^{(1)}$	$\boldsymbol{\epsilon} + \mathbf{0} + \mathbf{1}$

$$R_{ij}^{(2)} = R_{ij}^{(1)} + R_{i2}^{(1)}\left(R_{22}^{(1)}\right)^* R_{2j}^{(1)}$$

	By direct substitution
$R_{11}^{(2)}$	$\mathbf{1}^* + \mathbf{1}^*\mathbf{0}(\boldsymbol{\epsilon} + \mathbf{0} + \mathbf{1})^*\emptyset$
$R_{12}^{(2)}$	$\mathbf{1}^*\mathbf{0} + \mathbf{1}^*\mathbf{0}(\boldsymbol{\epsilon} + \mathbf{0} + \mathbf{1})^*(\boldsymbol{\epsilon} + \mathbf{0} + \mathbf{1})$
$R_{21}^{(2)}$	$\emptyset + (\boldsymbol{\epsilon} + \mathbf{0} + \mathbf{1})(\boldsymbol{\epsilon} + \mathbf{0} + \mathbf{1})^*\emptyset$
$R_{22}^{(2)}$	$\boldsymbol{\epsilon} + \mathbf{0} + \mathbf{1} + (\boldsymbol{\epsilon} + \mathbf{0} + \mathbf{1})(\boldsymbol{\epsilon} + \mathbf{0} + \mathbf{1})^*(\boldsymbol{\epsilon} + \mathbf{0} + \mathbf{1})$

	By direct substitution
$R_{11}^{(2)}$	$\mathbf{1^* + 1^*0(\epsilon + 0 + 1)^*\emptyset}$
$R_{12}^{(2)}$	$\mathbf{1^*0 + 1^*0(\epsilon + 0 + 1)^*(\epsilon + 0 + 1)}$
$R_{21}^{(2)}$	$\mathbf{\emptyset + (\epsilon + 0 + 1)(\epsilon + 0 + 1)^*\emptyset}$
$R_{22}^{(2)}$	$\mathbf{\epsilon + 0 + 1 + (\epsilon + 0 + 1)(\epsilon + 0 + 1)^*(\epsilon + 0 + 1)}$

	Simplified
$R_{11}^{(2)}$	$\mathbf{1^*}$
$R_{12}^{(2)}$	$\mathbf{1^*0(0 + 1)^*}$
$R_{21}^{(2)}$	$\mathbf{\emptyset}$
$R_{22}^{(2)}$	$\mathbf{(0 + 1)^*}$

The final regex for A is

$$R_{12}^{(2)} = \mathbf{1^*0(0 + 1)^*}$$

Observations

There are n^3 expressions $R_{ij}^{(k)}$

Each inductive step grows the expression 4-fold

$R_{ij}^{(n)}$ could have size 4^n

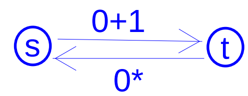
For all $\{i, j\} \subseteq \{1, \ldots, n\}$, $R_{ij}^{(k)}$ uses $R_{kk}^{(k-1)}$
so we have to write n^2 times the regex $R_{kk}^{(k-1)}$

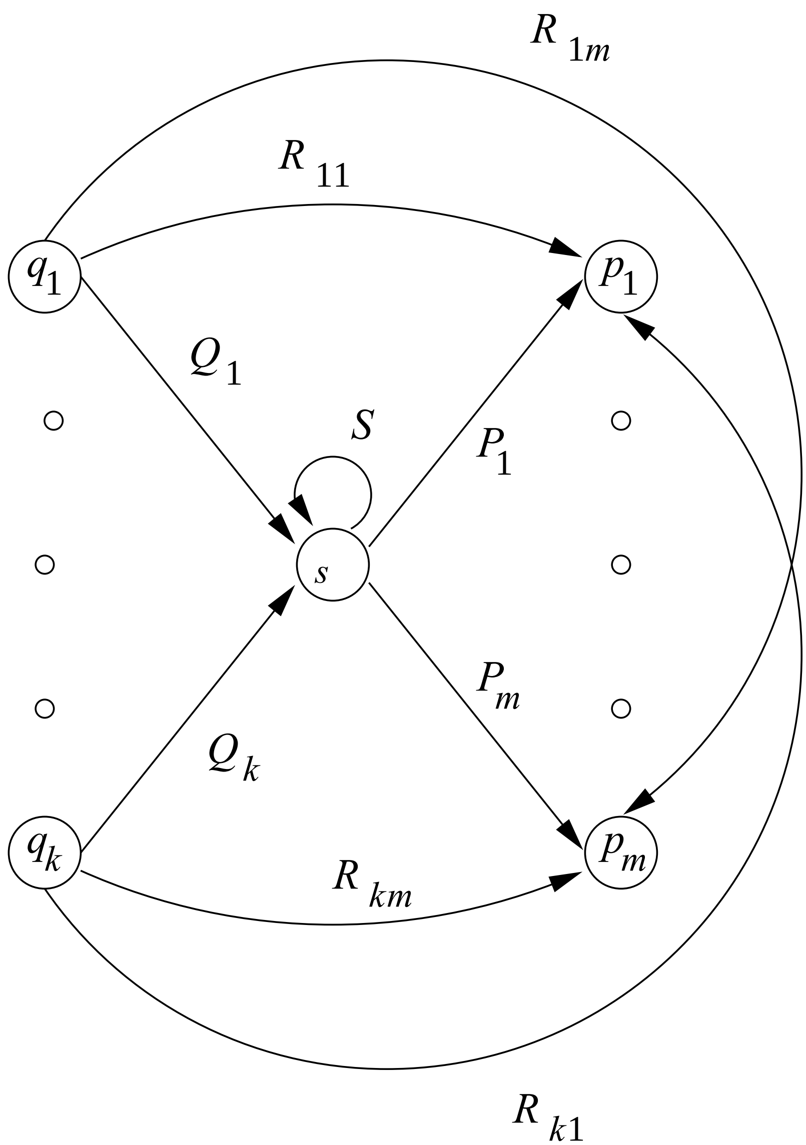
but most of them can be removed by annihilation!

We need a more efficient approach:
the state elimination technique

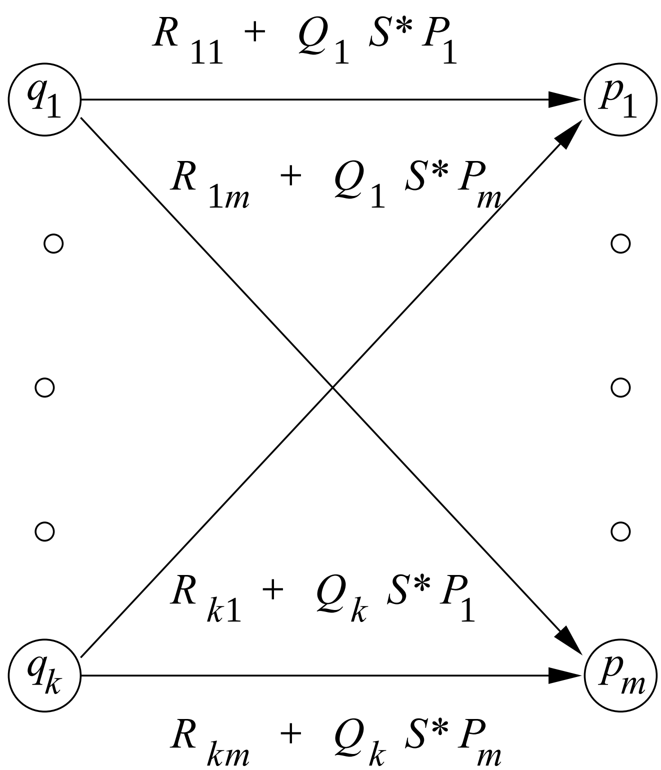
The state elimination technique

Let's label the edges with regex's instead of symbols

E.g., 

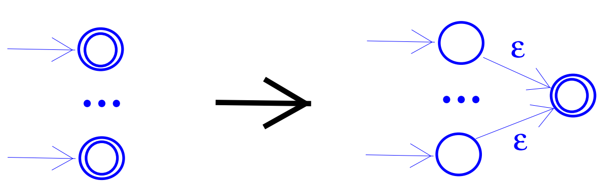


Now, let's eliminate state s.

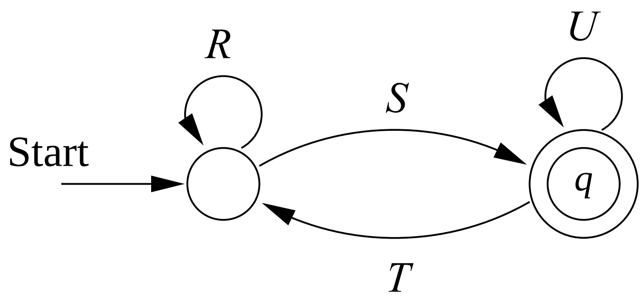


For each accepting state q, eliminate from the original automaton all states except q_0 and q.

or

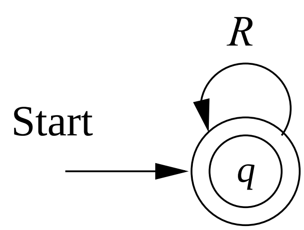


For each $q \in F$ we'll be left with an A_q that looks like



that corresponds to the regex $E_q = (R+SU^*T)^*SU^*$

or with A_q looking like



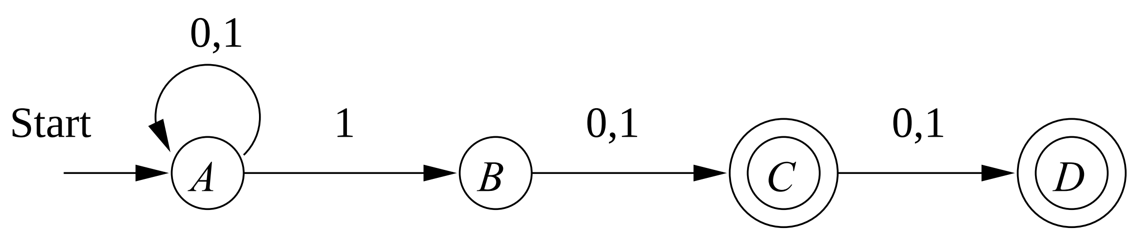
corresponding to the regex $E_q = R^*$

- The final expression is

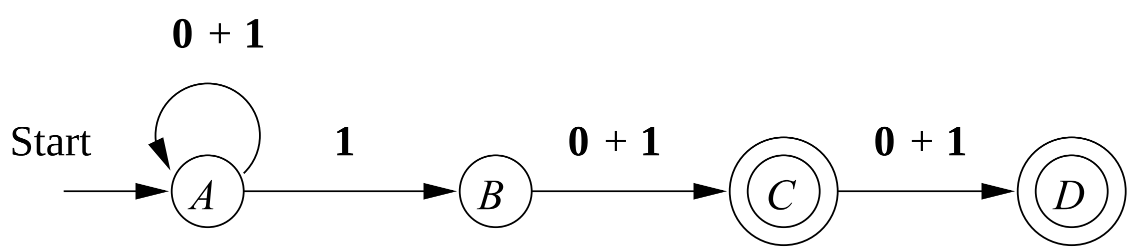
$$\bigoplus_{q \in F} E_q$$

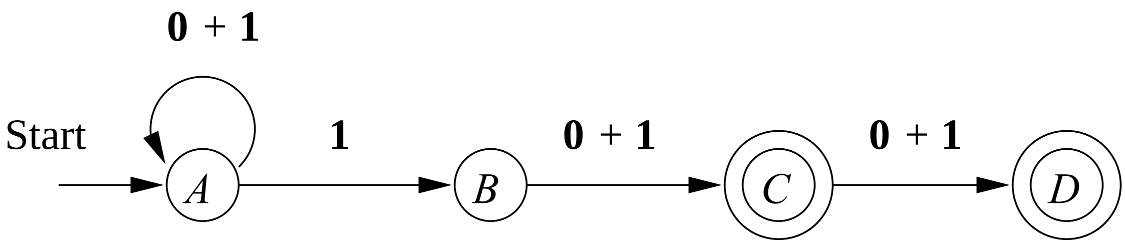
Note that the algorithm also works for NFAs and ε-NFAs.

Example: $\mathcal{A}$, where $L(\mathcal{A}) = \{W : w = x1b,$ or $w = x1bc,\ \ x \in \{0,1\}^*, \{b,c\} \subseteq \{0,1\}\}$

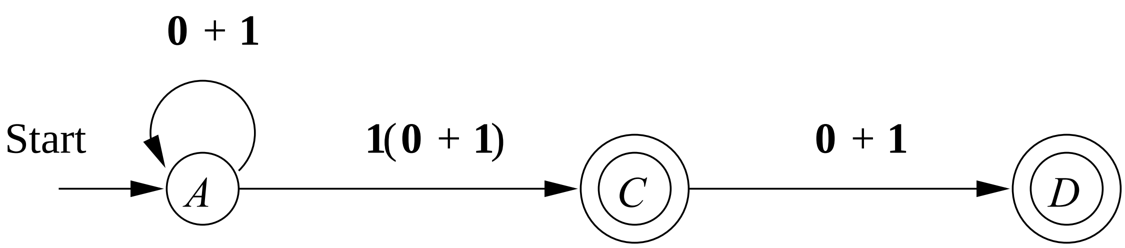


We turn this into an automaton with regex labels

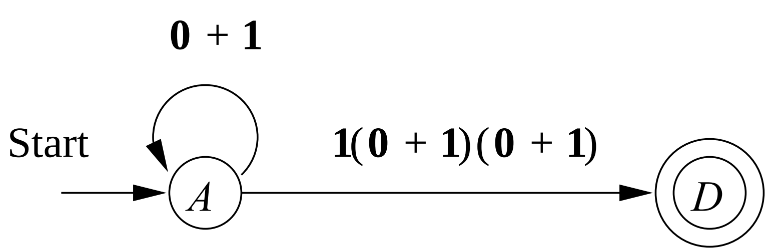




Let's eliminate state B

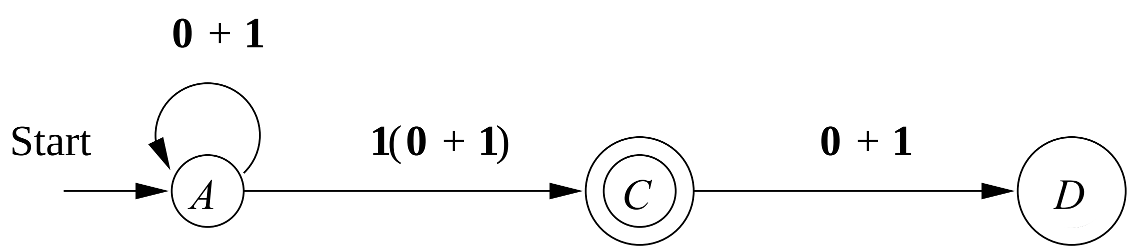


Then we eliminate state C and obtain $\mathcal{A}_D$

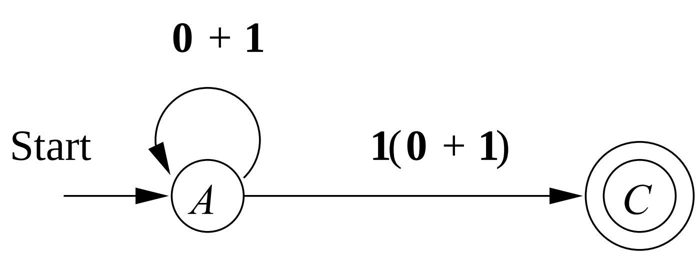


with regex $(\mathbf{0}+\mathbf{1})^*\mathbf{1}(\mathbf{0}+\mathbf{1})(\mathbf{0}+\mathbf{1})$

From



we can eliminate D to obtain $\mathcal{A}_C$



with regex $(\mathbf{0}+\mathbf{1})^*\mathbf{1}(\mathbf{0}+\mathbf{1})$

- The final expression is the sum of the previous two regex's:

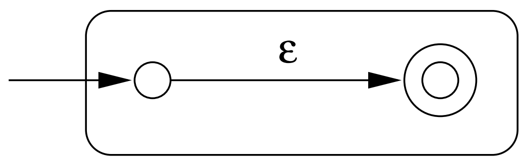
$$(\mathbf{0}+\mathbf{1})^*\mathbf{1}(\mathbf{0}+\mathbf{1})(\mathbf{0}+\mathbf{1})+(\mathbf{0}+\mathbf{1})^*\mathbf{1}(\mathbf{0}+\mathbf{1})$$

From regex's to ϵ-NFA's

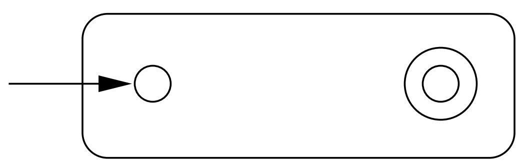
Theorem 3.7: For every regex R we can construct an ϵ-NFA A, s.t. $L(A) = L(R)$.

Proof: By structural induction:

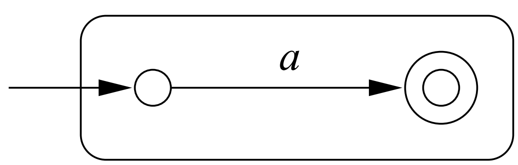
Basis: Automata for $\boldsymbol{\epsilon}$, $\emptyset$, and $\boldsymbol{a}$.



(a)



(b)



(c)

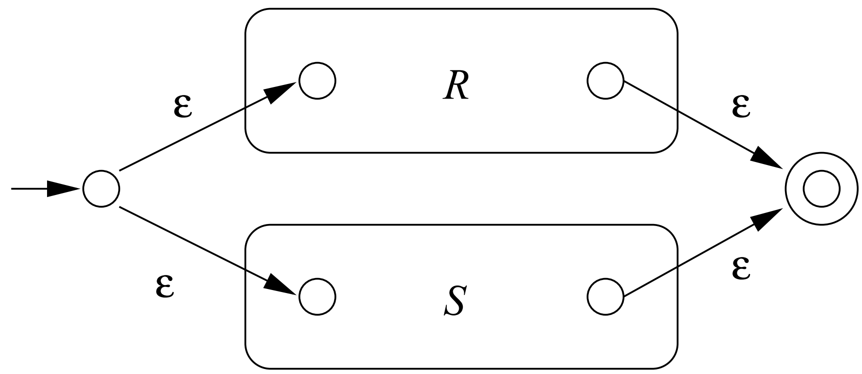
ε-NFAs with properties:

* unique start and final states

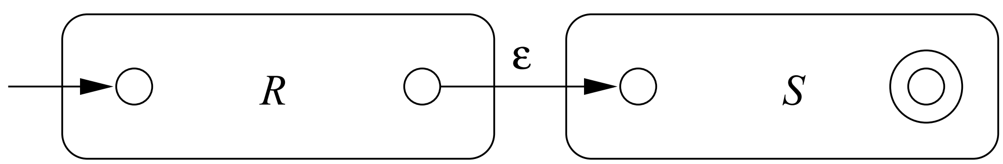
* no arcs into the start state

* no arcs out of the final state

Induction: Automata for $R + S$, RS, and R^*



(a)



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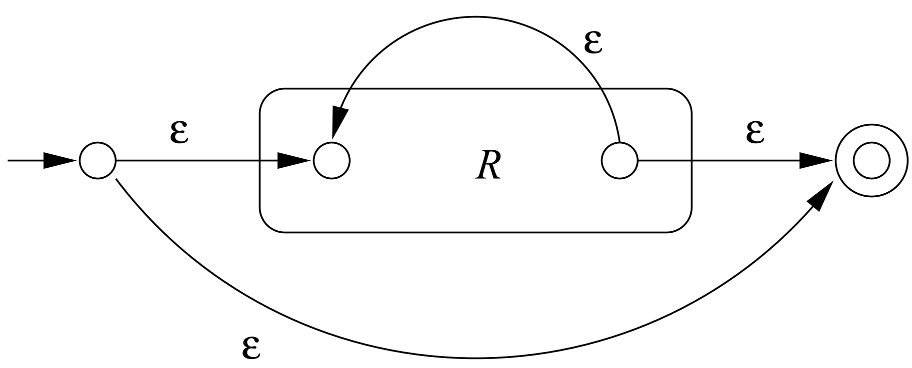
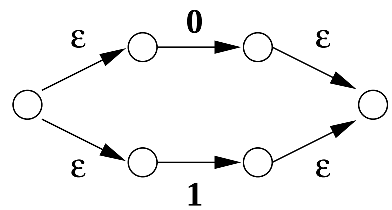
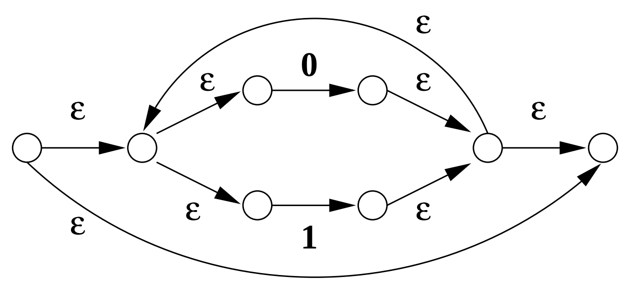


(c)

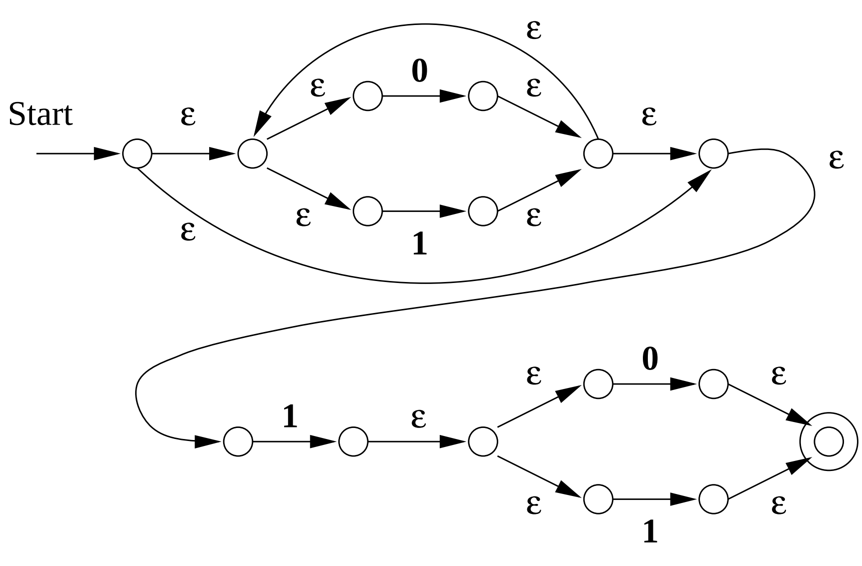
Example: We convert $(\mathbf{0}+\mathbf{1})^*\mathbf{1}(\mathbf{0}+\mathbf{1})$



(a)



(b)



(c)

It would be very useful if we could simplify regular languages/expressions and determine their properties.

Algebraic Laws for languages

- $L \cup M = M \cup L$.

Union is *commutative*.

- $(L \cup M) \cup N = L \cup (M \cup N)$.

Union is *associative*.

- $(LM)N = L(MN)$.

Concatenation is *associative*

Note: Concatenation is not commutative, *i.e.*, there are L and M such that $LM \neq ML$.

- $\emptyset \cup L = L \cup \emptyset = L$.

$\emptyset$ is *identity* for union.

- $\{\epsilon\}L = L\{\epsilon\} = L$.

$\{\epsilon\}$ is *left* and *right identity* for concatenation.

- $\emptyset L = L\emptyset = \emptyset$.

$\emptyset$ is *left* and *right annihilator* for concatenation.

- $L(M \cup N) = LM \cup LN$.

Concatenation is *left distributive* over union.

- $(M \cup N)L = ML \cup NL$.

Concatenation is *right distributive* over union.

- $L \cup L = L$.

Union is *idempotent*.

- $\emptyset^* = \{\epsilon\}$, $\{\epsilon\}^* = \{\epsilon\}$.

- $L^+ = LL^* = L^*L$, $L^* = L^+ \cup \{\epsilon\}$

- $(L^*)^* = L^*$. Closure is *idempotent*

Proof:

$$w \in (L^*)^* \iff w \in \bigcup_{i=0}^{\infty} \left(\bigcup_{j=0}^{\infty} L^j \right)^i$$

$$\iff \exists k, m_1, \ldots, m_k \in \mathbf{N} : \quad w = w_1 \ldots w_k \text{ with } w_1 \text{ in } L^{m1}, \ldots, w_k \text{ in } L^{mk}$$

$$\iff \exists p \in \mathbf{N} : w \in L^p \quad \boxed{\text{where } p = m_1 + \ldots + m_k}$$

$$\iff w \in \bigcup_{i=0}^{\infty} L^i$$

$$\iff w \in L^* \qquad \square$$

Claim. $(L \cup M)^* = (L^*M^*)^*$.

Proof. It is easy to see that $L \cup M$ is contained in L^*M^*, since L is contained in L^* which is contained in L^*M^*, and similarly M is contained in L^*M^*. Thus, the LHS is contained in the RHS.

To see that the RHS is also contained in the LHS, take any w in $(L^*M^*)^*$. Then, $w = w_1 w_2 \ldots w_n$, where each substring w_i is an element of L^*M^* and can thus be written as $x_{i1} \ldots x_{ik} y_{i1} \ldots y_{ih}$, where each sub-substring x_{ij} is an element of L and each y_{ij} an element of M. Thus, w is the concatenation of a sequence of strings, each of which is an element of $L \cup M$. Therefore, it is a string in $(L \cup M)^*$.

The above language laws all concern regex operations and can also be written as, e.g, L + M = M + L and L(M+N) = LM + LN.

Algebraic Laws for regex's

Evidently e.g. $(\mathbf{0}+\mathbf{1})\mathbf{1} = \mathbf{01}+\mathbf{11}$ because {0,1}{1} = {01,11}

Also e.g. $(\mathbf{00}+\mathbf{101})\mathbf{11} = \mathbf{0011}+\mathbf{10111}$.

More generally

$$(E+F)G = EG+FG$$

for any regex's E, F, and G or more generally, any languages E, F, and G.

- How do we verify that a general identity like above is true?

1. Prove it by hand.

2. Let the computer prove it.

In Chapter 4 we will learn how to test automatically if $E = F$, for any *concrete* regex's E and F, like $\mathbf{01} + \mathbf{11} = \mathbf{11} + \mathbf{01}$.

We want to test *general* identities, such as $\mathcal{E} + \mathcal{F} = \mathcal{F} + \mathcal{E}$, for *any* regex's $\mathcal{E}$ and $\mathcal{F}$.

or languages

Method: (The Test Technique!)

$\boxed{\text{symbol}}$

1. "Freeze" $\mathcal{E}$ to $\boldsymbol{a}_1$, and $\mathcal{F}$ to $\boldsymbol{a}_2$

2. Test automatically if the frozen identity is true, e.g. if $\boldsymbol{a}_1 + \boldsymbol{a}_2 = \boldsymbol{a}_2 + \boldsymbol{a}_1$

Question: Does this always work?

Answer: Yes, as long as the identities use only plus, dot, and star.

i.e. reg expr of language variables

Let's denote a generalized regex, such as $(\mathcal{E}+\mathcal{F})\mathcal{E}$ by

$$\mathsf{E}(\mathcal{E},\mathcal{F})$$

Now we can for instance make the substitution $\mathbf{S}=\{\mathcal{E}/\mathbf{0},\mathcal{F}/\mathbf{11}\}$ to obtain

$$\mathbf{S}\,(\mathsf{E}(\mathcal{E},\mathcal{F}))=(\mathbf{0}+\mathbf{11})\mathbf{0}$$

Theorem 3.13: Fix a "freezing" substitution $\spadesuit = \{\mathcal{E}_1/\boldsymbol{a}_1, \mathcal{E}_2/\boldsymbol{a}_2, \ldots, \mathcal{E}_m/\boldsymbol{a}_m\}$.

Let $\mathsf{E}(\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_m)$ be a generalized regex. Then for any regex's $E_1, E_2, \ldots, E_m$,

or languages

$$w \in L(\mathsf{E}(E_1, E_2, \ldots, E_m))$$

if and only if there are strings $w_i \in L(E_{ji})$, s.t.

$$w = w_1 w_2 \cdots w_k$$

and

$$a_{j_1} a_{j_2} \cdots a_{j_k} \in L(\mathsf{E}(\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_m))$$

Or, we "think" of each regular expr variable $\mathcal{E}_i$ as a symbol a_i.

Informally, to obtain w, we can first pick $a_{j1}\ a_{j2} \ldots a_{jk}$ in $L(E(a_1,a_2,\ldots,a_m))$ and then substitute for each a_{ji} any string from $L(E_{ji})$.

For example, suppose $E(\varepsilon_1,\varepsilon_2) = (\varepsilon_1 + \varepsilon_2)^*$. Then string w is in $L((E_1+E_2)^*)$ iff $w = w_1\ w_2 \ldots w_k$ such that $a_{j1}\ a_{j2} \ldots a_{jk}$ is in $L((a_1 + a_2)^*)$ and w_i is in $L(E_{ji})$.

For example: Suppose the alphabet is $\{1, 2\}$. Let $\mathsf{E}(\mathcal{E}_1, \mathcal{E}_2)$ be $(\mathcal{E}_1 + \mathcal{E}_2)\mathcal{E}_1$, and let E_1 be $\mathbf{1}$, and E_2 be $\mathbf{2}$. Then

$$w \in L(\mathsf{E}(E_1, E_2)) = L((E_1 + E_2)E_1) =$$

$$(\{1\} \cup \{2\})\{1\} = \{11\ ,\ 21\}$$

if and only if

$$\exists w_1 \in L(E_{j1}) \quad , \exists w_2 \in L(E_{j2}) \quad : \ w = w_1 w_2$$

and

$$a_{j_1} a_{j_2} \in L(\mathsf{E}(\boldsymbol{a}_1, \boldsymbol{a}_2))) = L((\boldsymbol{a}_1 + \boldsymbol{a}_2)\boldsymbol{a}_1) = \{a_1 a_1, a_2 a_1\}$$

if and only if

$j_1 = j_2 = 1$, or $j_1 = 2$, and $j_2 = 1$

In other words, w_1 is in $L(E_1) \cup L(E_2) = \{1,2\}$ and w_2 is in $L(E_1) = \{2\}$.

Another example, suppose E_1 = 1* and E_2 = 2*. Then
$L_0 = L((E_1 + E_2)E_1) = L((1^* + 2^*)1^*) = L(1^* + 2^*1^*)$.
$L((a_1 + a_2)a_1) = \{a_1\, a_1 + a_2\, a_1\}$.

String w is in L_0 iff there exist w_1 in $L(E_{j1})$ and w_2 in $L(E_{j2})$ such that $w = w_1\, w_2$ and $a_{j1}\, a_{j2}$ is in $\{a_1\, a_1 + a_2\, a_1\}$.

See page 120 of the textbook.

Proof of Theorem 3.13: We do a structural induction of E.

Basis: If E $= \boldsymbol{\epsilon}$, the frozen expression is also $\boldsymbol{\epsilon}$.

If E $= \emptyset$, the frozen expression is also $\emptyset$.

If E $= \mathcal{E}_1$, the frozen expression is $\boldsymbol{a}_1$. Now

$w \in L(\mathsf{E}(E_1))$ if and only if
w is in $L(E_1)$, since $L(\mathrm{E}(\boldsymbol{a}_1)) = \{\boldsymbol{a}_1\}$.

Induction:

Case 1: E = F + G.

Then ♠(E) = ♠(F) + ♠(G), and
L(♠(E)) = L(♠(F)) ∪ L(♠(G))

concrete or languages

Let F' and and G' be regex's. Then $w \in L(F' + G')$ if and only if $w \in L(F')$ or $w \in L(G')$.

Also, a string u is in E($\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$) iff it is in F($\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$) or in G($\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$). See the book for the rest of the proof using the I.H.

Case 2: E = F.G.

Then ♠(E) = ♠(F).♠(G), and
L(♠(E)) = L(♠(F)).L(♠(G))

concrete or languages

Let F' and and G' be regex's. Then $w \in L(F'.G')$ if and only if $w = w_1 w_2$, $w_1 \in L(F')$ and $w_2 \in L(G')$.

Also, a string u is in E($\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$) iff $u = u_1 u_2$ where u_1 is in F($\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$) and u_2 is in G($\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$). The rest is similar to the above case.

Case 3: E = F*.

Prove this case at home.

The test wouldn't work if the operation intersection were included in the regular expressions. E.g. consider $\mathcal{E} \wedge \mathcal{F} = \phi$.

The test for regular expressions and languages

Examples:

To prove $(\mathcal{L}+\mathcal{M})^* = (\mathcal{L}^*\mathcal{M}^*)^*$ it is enough to determine if $(\boldsymbol{a}_1+\boldsymbol{a}_2)^*$ is equivalent to $(\boldsymbol{a}_1^*\boldsymbol{a}_2^*)^*$

To verify $\mathcal{L}^* = \mathcal{L}^*\mathcal{L}^*$ test if $\boldsymbol{a}_1^*$ is equivalent to $\boldsymbol{a}_1^*\boldsymbol{a}_1^*$.

Question: Does $\mathcal{L}+\mathcal{ML} = (\mathcal{L}+\mathcal{M})\mathcal{L}$ hold?

To prove $(a_1 + a_2)^* == (a_1{}^* a_2{}^*)^*$, we first notice that L($(a_1{}^* a_2{}^*)^*$) is a subset of $L((a_1 + a_2)^*)$ because L($(a_1 + a_2)^*$) = (L($a_1 + a_2$))* = $\{a_1, a_2\}^*$ is the universe over $\{a_1, a_2\}$.

Since both a_1 *and* a_2 (as strings) are contained in $L(a_1{}^* a_2{}^*)$, $L(a_1 + a_2)$ is a subset of $L(a_1{}^* a_2{}^*)$, and hence $L((a_1 + a_2)^*)$ is a subset of L($(a_1{}^* a_2{}^*)^*$).

Does $a + ba = (a + b)a$ hold?

Theorem 3.14: $\mathsf{E}(\mathcal{E}_1, \ldots, \mathcal{E}_m) = \mathsf{F}(\mathcal{E}_1, \ldots, \mathcal{E}_m) \Leftrightarrow L(\spadesuit(\mathsf{E})) = L(\spadesuit(\mathsf{F}))$

Proof:

(Only if direction) $\mathsf{E}(\mathcal{E}_1, \ldots, \mathcal{E}_m) = \mathsf{F}(\mathcal{E}_1, \ldots, \mathcal{E}_m)$ means that $L(\mathsf{E}(E_1, \ldots, E_m)) = L(\mathsf{F}(E_1, \ldots, E_m))$ for any concrete regex's $E_1, \ldots, E_m$. In partic- or languages ular then $L(\spadesuit(\mathsf{E})) = L(\spadesuit(\mathsf{F}))$

or languages

(If direction) Let $E_1, \ldots, E_m$ be concrete regex's. Suppose $L(\spadesuit(\mathsf{E})) = L(\spadesuit(\mathsf{F}))$. Then by Theorem 3.13,

$$w \in L(\mathsf{E}(E_1, \ldots E_m)) \Leftrightarrow$$

$$\exists w_i \in L(E_i), w = w_{j_1} \cdots w_{j_m}, a_{j_1} \cdots a_{j_m} \in L(\spadesuit(\mathsf{E})) \Leftrightarrow$$

$$\exists w_i \in L(E_i), w = w_{j_1} \cdots w_{j_m}, a_{j_1} \cdots a_{j_m} \in L(\spadesuit(\mathsf{F})) \Leftrightarrow$$

$$w \in L(\mathsf{F}(E_1, \ldots E_m))$$

See page 121 of the textbook.

Properties of Regular Languages

- *Pumping Lemma.* Every regular language satisfies the pumping lemma. If somebody presents you with fake regular language, use the pumping lemma to show a contradiction.

- *Closure properties.* Building automata from components through operations, e.g. given L and M we can build an automaton for $L \cap M$.

- *Decision properties.* Computational analysis of automata, e.g. are two automata equivalent.

- *Minimization techniques.* We can save money since we can build smaller machines.

The Pumping Lemma Informally

Suppose $L_{01} = \{0^n 1^n : n \geq 1\}$ were regular.

Then it would be recognized by some DFA A, with, say, k states.

Let A read 0^k. On the way it will travel as follows:

ϵ	p_0
0	p_1
00	p_2
$\ldots$	$\ldots$
0^k	p_k

$\Rightarrow \exists i < j : p_i = p_j$ Call this state q.

Now you can fool A:

> If $\widehat{\delta}(q, 1^i) \in F$ the machine will foolishly accept $0^j 1^i$.
>
> If $\widehat{\delta}(q, 1^i) \notin F$ the machine will foolishly reject $0^i 1^i$.

Therefore L_{01} cannot be regular.

- Let's generalize the above reasoning.

Theorem 4.1.

The Pumping Lemma for Regular Languages.

Let L be regular.

Then $\exists n, \forall w \in L : |w| \geq n \Rightarrow w = xyz$ for some strings x, y and z such that

1. $y \neq \epsilon$

2. $|xy| \leq n$

3. $\forall k \geq 0,\ xy^k z \in L$

Proof: Suppose L is regular

Then L is recognized by some DFA A with, say, n states.

Let $w = a_1 a_2 \ldots a_m \in L,\ m >= n$.

Let $p_i = \widehat{\delta}(q_0, a_1 a_2 \cdots a_i)$.

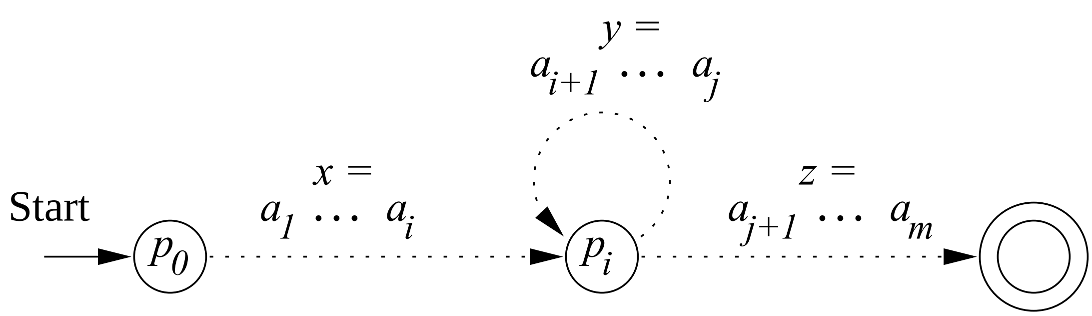
$\Rightarrow \exists i < j : p_i = p_j$, j <= n

Now $w = xyz$, where

1. $x = a_1 a_2 \cdots a_i$

2. $y = a_{i+1} a_{i+2} \cdots a_j$

3. $z = a_{j+1} a_{j+2} \ldots a_m$



Evidently $xy^k z \in L$, for any $k \geq 0$. $Q.E.D.$

Example: Let L_{eq} be the language of strings with equal number of zero's and one's.

Suppose L_{eq} is regular. Pick $w = 0^n 1^n \in L$.

By the pumping lemma $w = xyz$ for some strings x,y,z with $|xy| \leq n$, $y \neq \epsilon$ and $xy^k z \in L_{eq}$

$$w = \underbrace{000\cdots}_{x}\underbrace{\cdots 0}_{y}\underbrace{0111\cdots 11}_{z}$$

In particular, $xz \in L_{eq}$ (supposedly), but xz has fewer 0's than 1's.

L = $\{0^i 1^j \mid i > j\}$
Consider string w = $0^{n+1} 1^n$.
By the pumping lemma, we can partition w as w = xyz such that |xy| <= n, y <> ε, and $xy^k z$ in L.
But xz = $0^{n+1-|y|} 1^n$ is not in L.

Suppose $L_{pr} = \{1^p : p \text{ is prime }\}$ were regular.

Let n be given by the pumping lemma.

Choose a prime $p \geq n + 2$.

$$w = \overbrace{\underbrace{111\cdots}_{x}\underbrace{\cdots 1}_{\substack{y\\|y|=m}}\underbrace{1111\cdots 11}_{z}}^{p} \quad \text{with } |y| > 0 \text{ and } |xy| \leq n$$

Now, is $xy^{p-m}z \in L_{pr}$?

$|xy^{p-m}z| = |xz| + (p-m)|y| =$
$p - m + (p-m)m = (1+m)(p-m)$
which is not prime unless one of the factors is 1.

- $y \neq \epsilon \Rightarrow 1 + m > 1$

- $m = |y| \leq |xy| \leq n, \;\; p \geq n + 2$
$\Rightarrow p - m \geq n + 2 - n = 2.$

Closure Properties of Regular Languages

Let L and M be regular languages. Then the following languages are all regular:

- *Union:* $L \cup M$
- *Intersection:* $L \cap M$
- *Complement:* $\overline{N}$
- *Difference:* $L \setminus M$ (also L - M)
- *Reversal:* $L^R = \{w^R : w \in L\}$
- *Closure:* L^*.
- *Concatenation:* $L{\cdot}M$
- *Homomorphism:* $h(L) = \{h(w) : w \in L, h$ is a homom. $\}$

  $\boxed{h(a_1\ a_2\ \ldots\ a_n) = h(a_1)h(a_2)\ldots h(a_n)}$
- *Inverse homomorphism:* $h^{-1}(L) = \{w \in \Sigma : h(w) \in L, h : \Sigma \to \Delta^*$ is a homom. $\}$

Theorem 4.4. For any regular L and M, $L \cup M$ is regular.

Proof. Let $L = L(E)$ and $M = L(F)$. Then $L(E + F) = L \cup M$ by definition.

Theorem 4.5. If L is a regular language over Σ, then so is $\overline{L} = \Sigma^* \setminus L$.

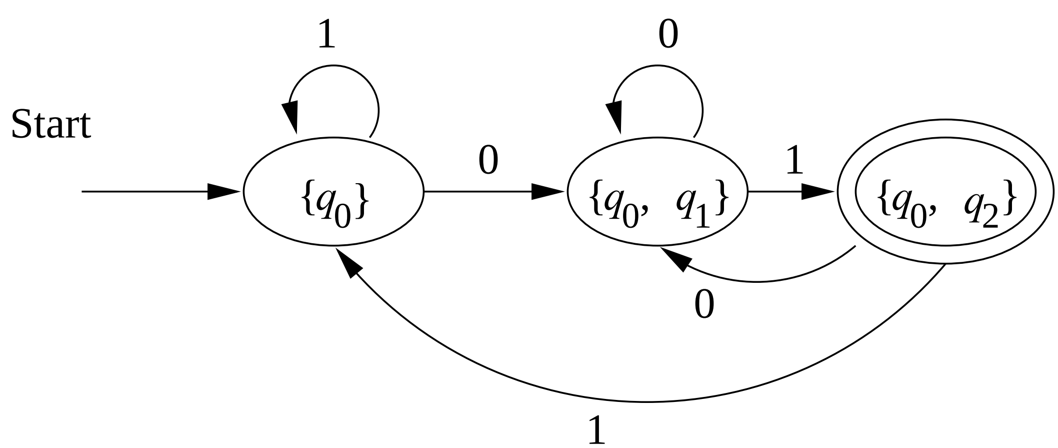
Proof. Let L be recognized by a DFA

$$A = (Q, \Sigma, \delta, q_0, F).$$

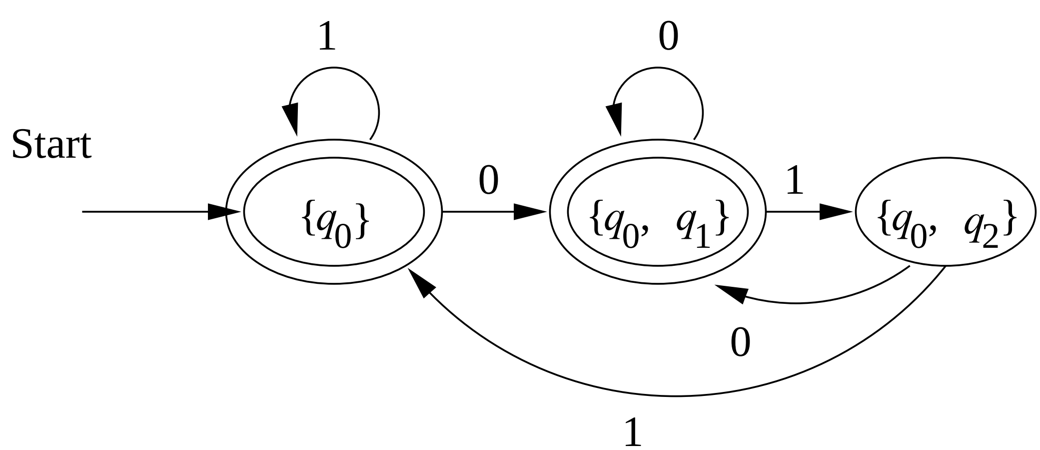
Let $B = (Q, \Sigma, \delta, q_0, Q \setminus F)$. Now $L(B) = \overline{L}$.

Example:

Let L be recognized by the DFA below



Then $\overline{L}$ is recognized by



Question: What are the regex's for L and $\overline{L}$

Theorem 4.8. If L and M are regular, then so is $L \cap M$.

Proof. By DeMorgan's law $L \cap M = \overline{\overline{L} \cup \overline{M}}$. We already that regular languages are closed under complement and union.

We shall also give a nice direct proof, the *Cartesian* construction from the e-commerce example.

Theorem 4.8. If L and M are regular, then so is $L \cap M$.

Proof. Let L be the language of

$$A_L = (Q_L, \Sigma, \delta_L, q_L, F_L)$$

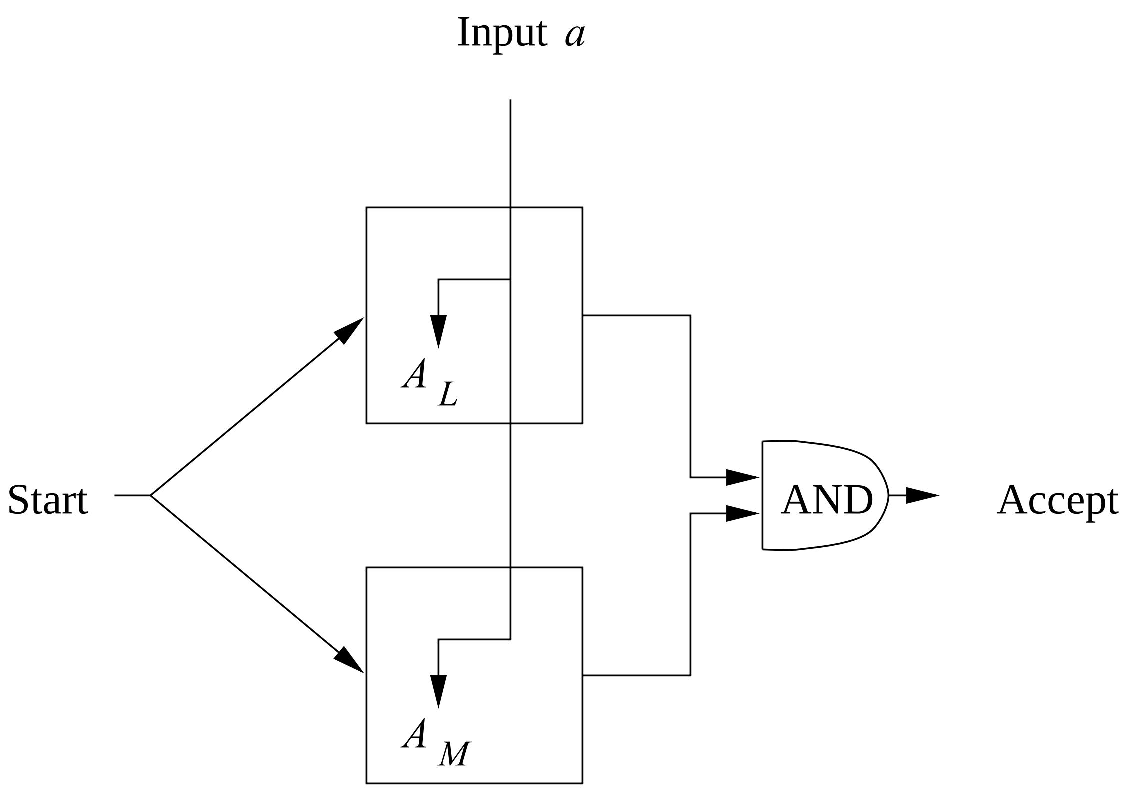
and M be the language of

$$A_M = (Q_M, \Sigma, \delta_M, q_M, F_M)$$

We assume w.l.o.g. that both automata are deterministic.

We shall construct an automaton that simulates A_L and A_M in parallel, and accepts if and only if both A_L and A_M accept.

If A_L goes from state p to state s on reading a, and A_M goes from state q to state t on reading a, then $A_{L \cap M}$ will go from state (p, q) to state (s, t) on reading a.



Formally

$$A_{L\cap M} = (Q_L \times Q_M, \Sigma, \delta_{L\cap M}, (q_L, q_M), F_L \times F_M),$$

where

$$\delta_{L\cap M}((p,q),a) = (\delta_L(p,a), \delta_M(q,a))$$

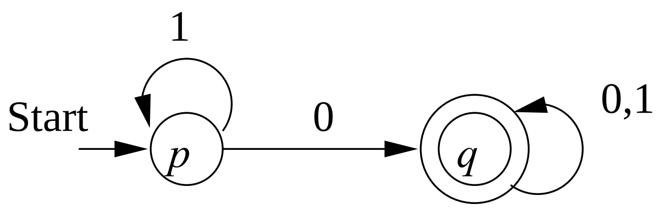
It will be shown in the tutorial by an induction on $|w|$ that

$$\hat{\delta}_{L\cap M}((q_L,q_M),w) = \left(\hat{\delta}_L(q_L,w), \hat{\delta}_M(q_M,w)\right)$$

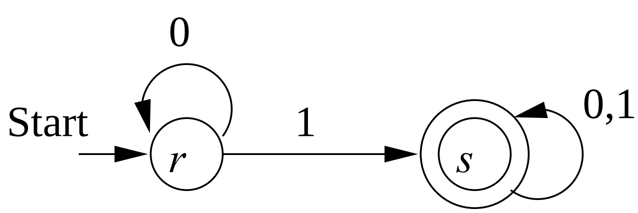
The claim then follows.

Question: Why?

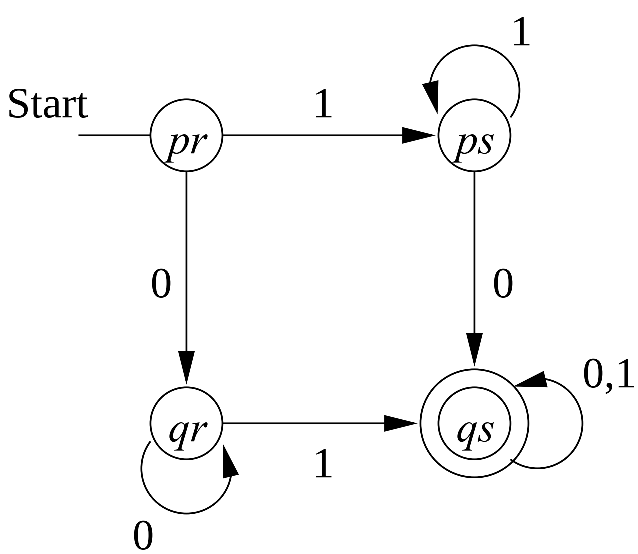
Example: $(c) = (a) \times (b)$



(a)



(b)



(c)

Another example?
{Binary strings that begin with 1 and
represent numbers divisible by 3}

Theorem 4.10. If L and M are regular languages, then so is $L \setminus M$. (Also denoted as L - M.)

Proof. Observe that $L \setminus M = L \cap \overline{M}$. We already know that regular languages are closed under complement and intersection.

Theorem 4.11. If L is a regular language, then so is L^R.

Proof 1: Let L be recognized by an FA A. Turn A into an FA for L^R, by

1. Reversing all arcs.
2. Make the old start state the new sole accepting state.
3. Create a new start state p_0, with $\delta(p_0, \epsilon) = F$ (the old accepting states).

Theorem 4.11. If L is a regular language, then so is L^R.

Proof 2: Let L be described by a regex E. We shall construct a regex E^R, such that $L(E^R) = (L(E))^R$.

We proceed by a structural induction on E.

Basis: If E is $\boldsymbol{\epsilon}$, $\emptyset$, or $\boldsymbol{a}$, then $E^R = E$.

Induction:

1. $E = F + G$. Then $E^R = F^R + G^R$
2. $E = F.G$. Then $E^R = G^R.F^R$
3. $E = F^*$. Then $E^R = (F^R)^*$

We will show by structural induction on E on blackboard in class that

$$L(E^R) = (L(E))^R$$

Homomorphisms

A *homomorphism* on Σ is a function $h : \Sigma \rightarrow \Theta^*$, where Σ and Θ are alphabets.

Let $w = a_1 a_2 \cdots a_n \in \Sigma^*$. Then

$$h(w) = h(a_1)h(a_2)\cdots h(a_n)$$

and

$$h(L) = \{h(w) : w \in L\}$$

Example: Let $h : \{0,1\}^* \rightarrow \{a,b\}^*$ be defined by $h(0) = ab$, and $h(1) = \epsilon$. Now $h(0011) = abab$.

Example: $h(L(\mathbf{10^*1})) = L((\boldsymbol{ab})^*)$.

Theorem 4.14: $h(L)$ is regular, whenever L is.

Proof: E.g., h(0*1+(0+1)*0) = h(0)*h(1)+(h(0)+h(1))*h(0)

Let $L = L(E)$ for a regex E. We claim that $L(h(E)) = h(L)$.

Basis: If E is $\boldsymbol{\epsilon}$ or $\emptyset$. Then $h(E) = E$, and $L(h(E)) = L(E) = h(L(E))$.

If E is $\boldsymbol{a}$, then $L(E) = \{a\}$, $L(h(E)) = L(h(\boldsymbol{a})) = \{h(a)\} = h(L(E))$.

Induction:

Case 1: $G = E + F$. Now $L(h(E + F)) = L(h(E)+h(F)) = L(h(E)) \cup L(h(F)) = h(L(E)) \cup h(L(F)) = h(L(E) \cup L(F)) = h(L(E + F))$.

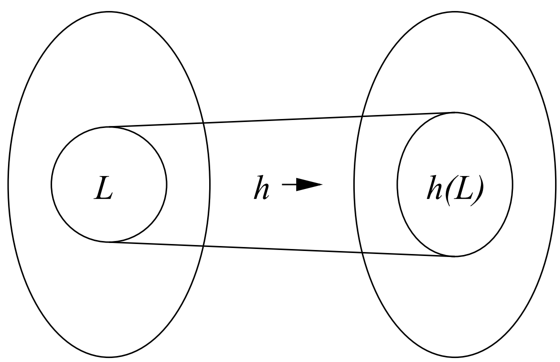
Case 2: $G = E.F$. Now $L(h(E.F)) = L(h(E)).L(h(F)) = h(L(E)).h(L(F)) = h(L(E).L(F)) = h(L(E.F))$

Case 3: $G = E^*$. Now $L(h(E^*)) = L(h(E)^*) = L(h(E))^* = h(L(E))^* = h(L(E)^*) = h(L(E^*))$

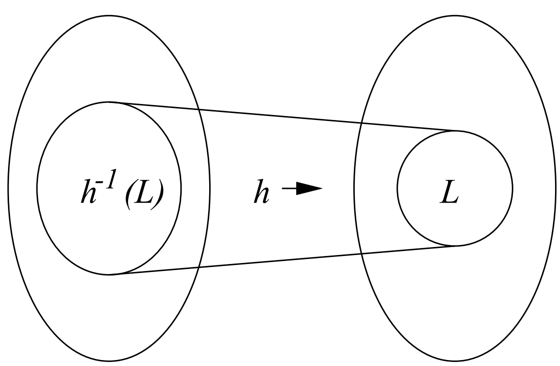
Inverse Homomorphism

Let $h : \Sigma \rightarrow \Theta^*$ be a homom. Let $L \subseteq \Theta^*$, and define

$$h^{-1}(L) = \{w \in \Sigma^* : h(w) \in L\}$$



(a)



(b)

Example: Let $h : \{a, b\} \to \{0, 1\}^*$ be defined by $h(a) = 01$, and $h(b) = 10$. If $L = L((\mathbf{00}+\mathbf{1})^*)$, then $h^{-1}(L) = L((\boldsymbol{ba})^*)$.

Claim: $h(w) \in L$ if and only if $w = (ba)^n$

Proof: Let $w = (ba)^n$. Then $h(w) = (1001)^n \in L$.

Let $h(w) \in L$, and suppose $w \notin L((\boldsymbol{ba})^*)$. There are four cases to consider.

1. w begins with a. Then $h(w)$ begins with 01 and $\notin L((\mathbf{00}+\mathbf{1})^*)$.

2. w ends in b. Then $h(w)$ ends in 10 and $\notin L((\mathbf{00}+\mathbf{1})^*)$.

3. $w = xaay$. Then $h(w) = z0101v$ and $\notin L((\mathbf{00}+\mathbf{1})^*)$.

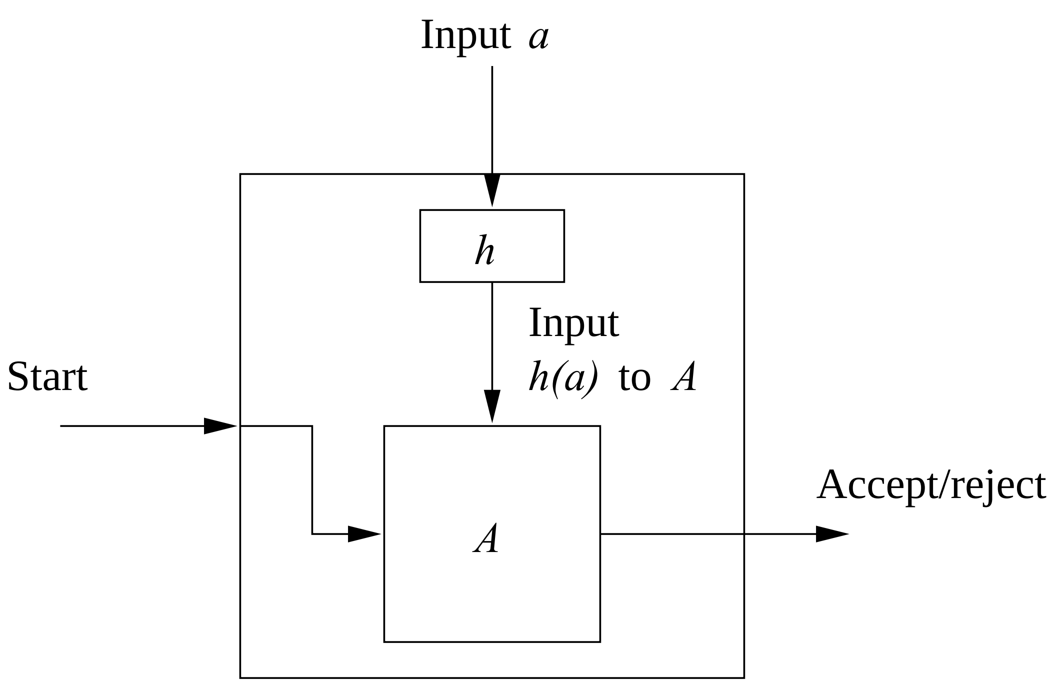
4. $w = xbby$. Then $h(w) = z1010v$ and $\notin L((\mathbf{00}+\mathbf{1})^*)$.

Theorem 4.16: Let $h : \Sigma \to \Theta^*$ be a homom., and $L \subseteq \Theta^*$ regular. Then $h^{-1}(L)$ is regular.

Proof: Let L be the language of $A = (Q, \Theta, \delta, q_0, F)$. We define $B = (Q, \Sigma, \gamma, q_0, F)$, where

$$\gamma(q, a) = \widehat{\delta}(q, h(a))$$

It will be shown by induction on $|w|$ in the tutorial that $\widehat{\gamma}(q_0, w) = \widehat{\delta}(q_0, h(w))$



Decision Properties

We consider the following:

1. Converting among representations for regular languages.

2. Is $L = \emptyset$? Is L finite?

3. Is $w \in L$?

4. Do two descriptions define the same language?

From NFA's to DFA's

Suppose the ϵ-NFA has n states.

To compute ECLOSE(p) we follow at most n^2 arcs.

The DFA has 2^n states, for each state S and each $a \in \Sigma$ we compute $\delta_D(S, a)$ in n^3 steps. Grand total is $O(n^3 2^n)$ steps.

If we compute δ for reachable states only, we need to compute $\delta_D(S, a)$ only s times, where s is the number of reachable states. Grand total is $O(n^3 s)$ steps.

From DFA to NFA

All we need to do is to put set brackets around the states. Total $O(n)$ steps.

From FA to regex

We need to compute n^3 entries of size up to 4^n. Total is $O(n^3 4^n)$.

The FA is allowed to be an NFA. If we first wanted to convert the NFA to a DFA, the total time would be doubly exponential

From regex to FA's We can build an expression tree for the regex in n steps.

We can construct the automaton in n steps.

Eliminating ϵ-transitions takes $O(n^3)$ steps.

If you want a DFA, you might need an exponential number of steps.

Testing emptiness

$L(A) \neq \emptyset$ for FA A if and only if a final state is reachable from the start state in A. Total $O(n^2)$ steps.

Alternatively, we can inspect a regex E and tell if $L(E) = \emptyset$. We use the following method:

$E = F + G$. Now $L(E)$ is empty if and only if both $L(F)$ and $L(G)$ are empty.

$E = F.G$. Now $L(E)$ is empty if and only if either $L(F)$ or $L(G)$ is empty.

$E = F^*$. Now $L(E)$ is never empty, since $\epsilon \in L(E)$.

$E = \epsilon$. Now $L(E)$ is not empty.

$E = \boldsymbol{a}$. Now $L(E)$ is not empty.

$E = \boldsymbol{\emptyset}$. Now $L(E)$ is empty.

Finiteness: How to decide if L(A) is finite for DFA A?

Testing membership

To test $w \in L(A)$ for DFA A, simulate A on w. If $|w| = n$, this takes $O(n)$ steps.

If A is an NFA and has s states, simulating A on w takes $O(ns^2)$ steps.

If A is an ϵ-NFA and has s states, simulating A on w takes $O(ns^3)$ steps.

If $L = L(E)$, for regex E of length s, we first convert E to an ϵ-NFA with $2s$ states. Then we simulate w on this machine, in $O(ns^3)$ steps.

Does $L((0+1)^*0(0+1)^3 1^*)$ contain 10101011 or 101011101?

Equivalence and Minimization of Automata

Let $A = (Q, \Sigma, \delta, q_0, F)$ be a DFA, and $\{p, q\} \subseteq Q$. We define

$$p \equiv q \;\Leftrightarrow\; \forall w \in \Sigma^* \; : \; \widehat{\delta}(p, w) \in F \text{ iff } \widehat{\delta}(q, w) \in F$$

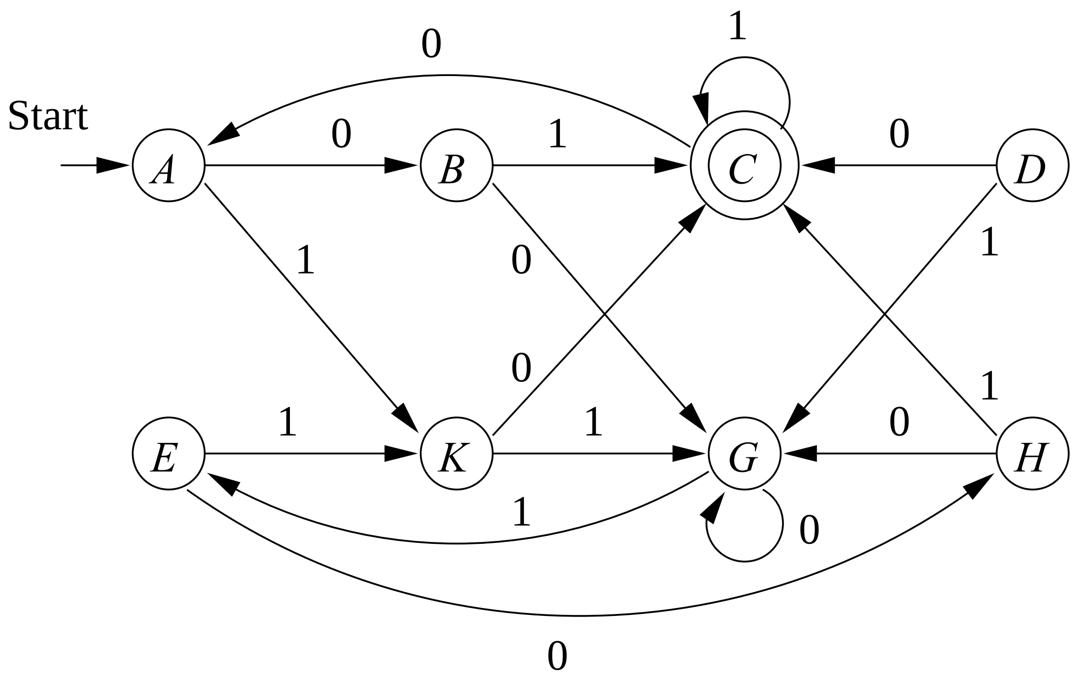
- If $p \equiv q$ we say that p and q are *equivalent*

- If $p \not\equiv q$ we say that p and q are *distinguishable*

IOW (in other words) p and q are distinguishable iff

$\exists w : \widehat{\delta}(p, w) \in F$ and $\widehat{\delta}(q, w) \notin F,$ or vice versa

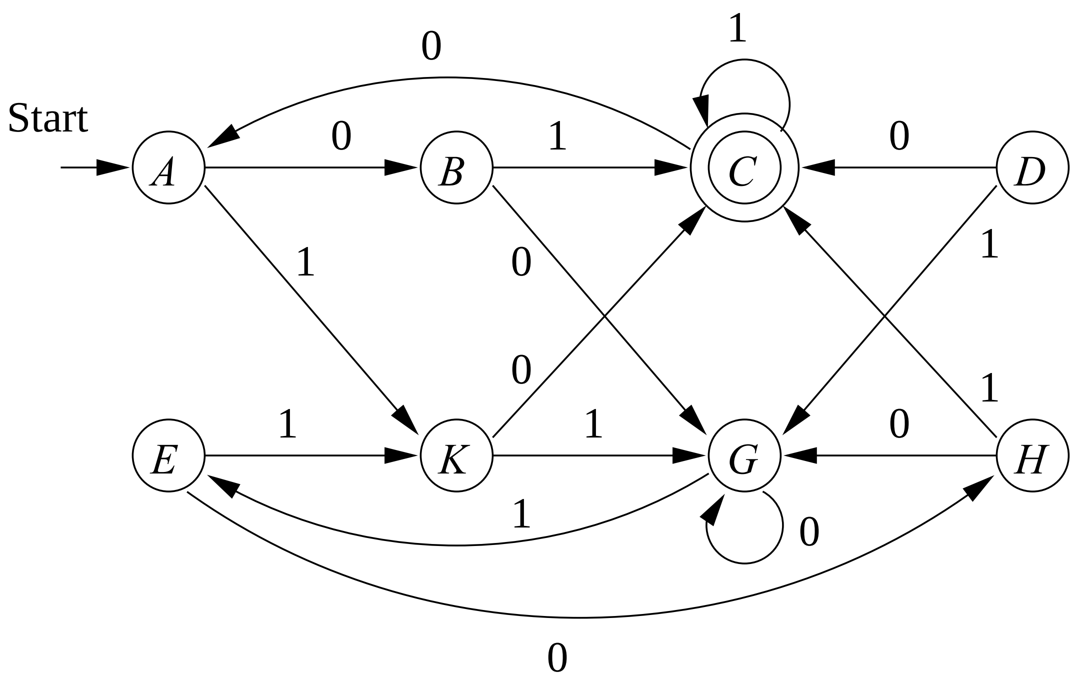
Example:



$\widehat{\delta}(C,\epsilon) \in F, \widehat{\delta}(G,\epsilon) \notin F \Rightarrow C \not\equiv G$

$\widehat{\delta}(A,01) = C \in F, \widehat{\delta}(G,01) = E \notin F \Rightarrow A \not\equiv G$

What about A and E?



$\widehat{\delta}(A, \epsilon) = A \notin F, \widehat{\delta}(E, \epsilon) = E \notin F$

$\widehat{\delta}(A, 1) = \mathrm{K} = \widehat{\delta}(E, 1)$

Therefore $\widehat{\delta}(A, 1x) = \widehat{\delta}(E, 1x) = \widehat{\delta}(\mathrm{K}, x)$

$\widehat{\delta}(A, 00) = G = \widehat{\delta}(E, 00)$

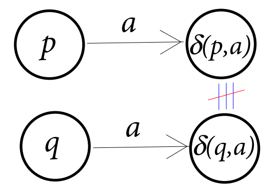
$\widehat{\delta}(A, 01) = C = \widehat{\delta}(E, 01)$

Conclusion: $A \equiv E$.

We can compute distinguishable pairs with the following inductive *table filling* (TF) *algorithm*:

Basis: If $p \in F$ and $q \notin F$, then $p \not\equiv q$.

Induction: If $\exists a \in \Sigma : \delta(p, a) \not\equiv \delta(q, a)$, then $p \not\equiv q$.



Example: Applying the table filling algo to A:

B	x						
C	x	x					
D	x	x	x				
E		x	x	x			
K	x	x	x		x		
G	x	x	x	x	x	x	
H	x		x	x	x	x	x
	A	B	C	D	E	K	G

Theorem 4.20: If p and q are not distinguished by the TF-algo, then $p \equiv q$.

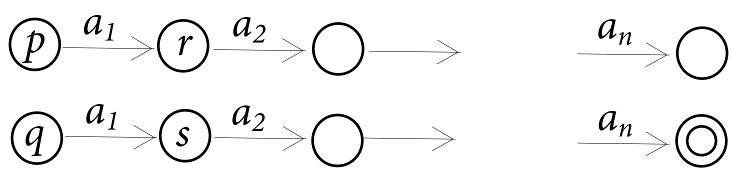
Proof: Suppose to the contrary that that there is a *bad pair* $\{p, q\}$, s.t.

1. $\exists w : \widehat{\delta}(p, w) \in F, \widehat{\delta}(q, w) \notin F$, or vice versa.

2. The TF-algo does not distinguish between p and q.

Let $w = a_1 a_2 \cdots a_n$ be the shortest string that identifies a bad pair $\{p, q\}$.

Now $w \neq \epsilon$ since otherwise the TF-algo would in the basis distinguish p from q. Thus $n \geq 1$.



Consider states $r = \delta(p, a_1)$ and $s = \delta(q, a_1)$. Now $\{r, s\}$ cannot be a bad pair since $\{r, s\}$ would be indentified by a string shorter than w. Therefore, the TF-algo must have discovered that r and s are distinguishable.

But then the TF-algo would distinguish p from q in the inductive part.

Thus there are no bad pairs and the theorem is true.

Testing Equivalence of Regular Languages

Let L and M be reg langs (each given in some form).

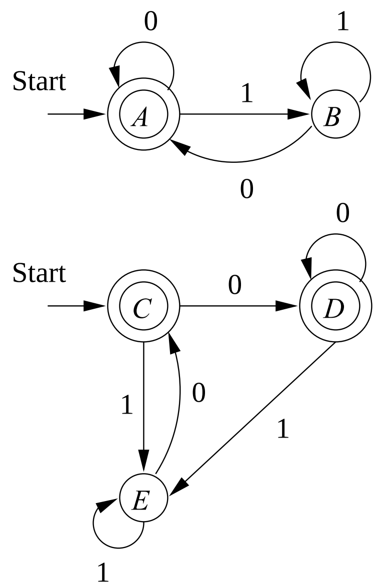
To test if $L = M$

1. Convert both L and M to DFA's.

2. Imagine a DFA that is the union of the two DFA's (never mind there are two start states)

3. If TF-algo says that the two start states are distinguishable, then $L \neq M$, otherwise $L = M$.

Example:



We can "see" that both DFA's accept $L(\epsilon + (\mathbf{0}+\mathbf{1})^*\mathbf{0})$. The result of the TF-algo is

	A	B	C	D
B	x			
C		x		
D		x		
E	x		x	x

Therefore the two automata are equivalent.

Minimization of DFA's

We can use the TF-algo to minimize a DFA by merging all equivalent states. IOW, replace each state p by $p/_{\equiv}$.

Example: The DFA on slide 119 has equivalence classes $\{\{A,E\},\{B,H\},\{C\},\{D,K\},\{G\}\}$.

The "union" DFA on slide 125 has equivalence classes $\{\{A,C,D\},\{B,E\}\}$.

Note: In order for $p/_{\equiv}$ to be an *equivalence class*, the relation $\equiv$ has to be an *equivalence relation* (reflexive, symmetric, and transitive).

Theorem 4.23: If $p \equiv q$ and $q \equiv r$, then $p \equiv r$.

Proof: Suppose to the contrary that $p \not\equiv r$. Then $\exists w$ such that $\hat{\delta}(p, w) \in F$ and $\hat{\delta}(r, w) \notin F$, or vice versa.

OTH, $\hat{\delta}(q, w)$ is either accpeting or not.

Case 1: $\hat{\delta}(q, w)$ is accepting. Then $q \not\equiv r$.

Case 2: $\hat{\delta}(q, w)$ is not accepting. Then $p \not\equiv q$.

The vice versa case is proved symmetrically

Therefore it must be that $p \equiv r$.

Assume A has no inaccessible states.

To minimize a DFA $A = (Q, \Sigma, \delta, q_0, F)$ construct a DFA $B = (Q/_{\equiv}, \Sigma, \gamma, q_0/_{\equiv}, F/_{\equiv})$, where

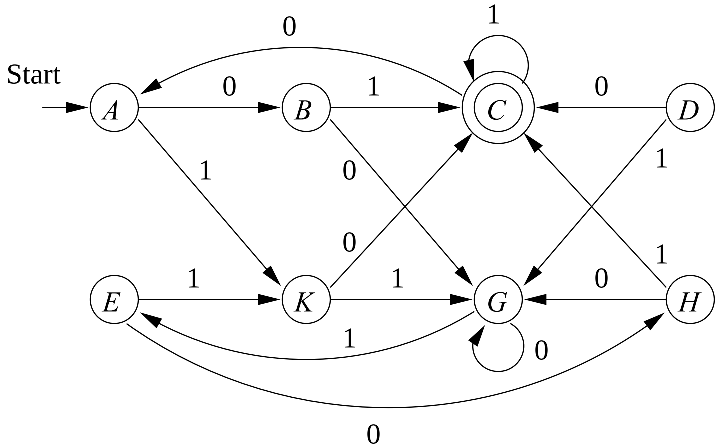
$$\gamma(p/_{\equiv}, a) = \delta(p, a)/_{\equiv}$$

In order for B to be well defined we have to show that

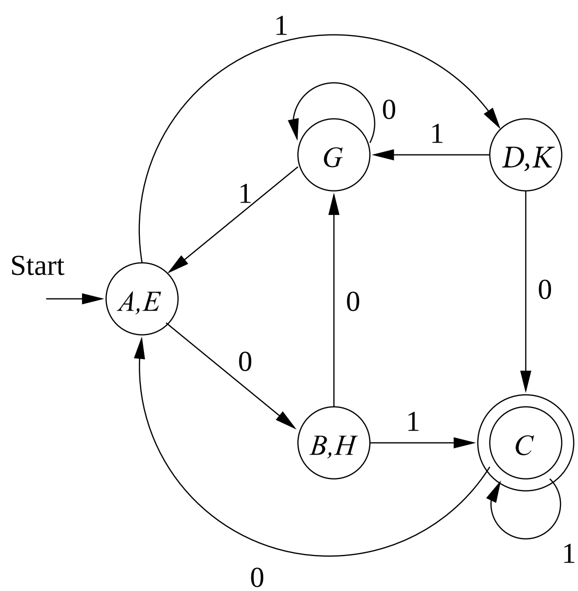
If $p \equiv q$ then $\delta(p, a) \equiv \delta(q, a)$

If $\delta(p, a) \not\equiv \delta(q, a)$, then the TF-algo would conclude $p \not\equiv q$, so B is indeed well defined. Note also that $F/_{\equiv}$ contains all and only the accepting states of A.

Example: We can minimize

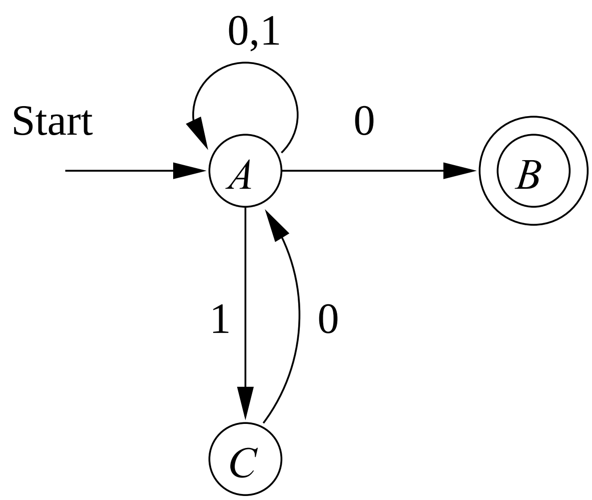


to obtain



NOTE: We cannot apply the TF-algo to NFA's.

For example, to minimize



we simply remove state C.

However, $A \not\equiv C$.

Why the Minimized DFA Can't Be Beaten

Let B be the minimized DFA obtained by applying the TF-algo to DFA A.

We already know that $L(A) = L(B)$.

What if there existed a DFA C, with $L(C) = L(B)$ and fewer states than B?

Then run the TF-algo on B "union" C.

Since $L(B) = L(C)$ we have $q_0^B \equiv q_0^C$.

Also, $\delta(q_0^B, a) \equiv \delta(q_0^C, a)$, for any a.

Claim: For each state p in B there is at least one state q in C, s.t. $p \equiv q$.

Proof of claim: There are no inaccessible states, so $p = \widehat{\delta}(q_0^B, a_1 a_2 \cdots a_k)$, for some string $a_1 a_2 \cdots a_k$. Now $q = \widehat{\delta}(q_0^C, a_1 a_2 \cdots a_k)$, and $p \equiv q$.

Since C has fewer states than B, there must be two states r and s of B such that $r \equiv t \equiv s$, for some state t of C. But then $r \equiv s$ (why?) which is a contradiction, since B was constructed by the TF-algo.

Context-Free Grammars and Languages

• We have seen that many languages cannot be regular. Thus we need to consider larger classes of langs.

• *Contex-Free Languages* (CFL's) played a cen-tral role in natural languages since the 1950's, and in compilers since the 1960's.

• *Context-Free Grammars* (CFG's) are the basis of BNF-syntax.

• Today CFL's are increasingly important for XML and their DTD's.

We'll look at: CFG's, the languages they generate, parse trees, pushdown automata, and closure properties of CFL's.

Pumping Lemma, decision property.

Informal example of CFG's

Consider $L_{pal} = \{w \in \Sigma^* : w = w^R\}$

For example $\texttt{otto} \in L_{pal}$, $\texttt{madamimadam} \in L_{pal}$.

In Finnish language e.g. $\texttt{saippuakauppias} \in L_{pal}$ ("soap-merchant")

DoGeeseSeeGod
NoMelonNoLemon
Zilliz

Let $\Sigma = \{0, 1\}$ and suppose L_{pal} were regular.

Let n be given by the pumping lemma. Then $w = 0^n 1 0^n \in L_{pal}$, and consider any partition $xyz = w$ with $|y| > 0$ and $|xy| <= n$. $xyyz$ is a contradiction.

Let's define L_{pal} inductively:

Basis: ϵ, 0, and 1 are palindromes.

Induction: If w is a palindrome, so are $0w0$ and $1w1$.

Circumscription: Nothing else is a palindrome.

CFGs provide a formal mechanism for definitions such as the one for L_{pal}.

1. $P \rightarrow \epsilon$
2. $P \rightarrow 0$
3. $P \rightarrow 1$
4. $P \rightarrow 0P0$
5. $P \rightarrow 1P1$

0 and 1 are *terminals*

P is a *variable* (or *nonterminal*, or *syntactic category*)

P is in this grammar also the *start symbol*.

1–5 are *productions* (or *rules*)

Some real examples from Sipser.

Formal definition of CFG's

A *context-free grammar* is a quadruple

$$G = (V, T, P, S)$$

where

V is a finite set of *variables* or *nonterminals*.

T is a finite set of *terminals*.

P is a finite set of *productions* of the form $A \to \alpha$, where A is a variable and $\alpha \in (V \cup T)^*$

S is a designated variable called the *start symbol*.

Example: $G_{pal} = (\{P\}, \{0, 1\}, A, P)$, where $A = \{P \to \epsilon, P \to 0, P \to 1, P \to 0P0, P \to 1P1\}$.

Sometimes we group productions with the same head, e.g. $A = \{P \to \epsilon|0|1|0P0|1P1\}$.

Example: Regular expressions over $\{0, 1\}$ can be defined by the grammar

$$G_{regex} = (\{E\}, \{\mathbf{0}, \mathbf{1}, +, \cdot, \phi, \varepsilon, *, (,)\}, A, E)$$

where $A =$

$$\{E \to \mathbf{0}, E \to \mathbf{1}, E \to E{\cdot}E, E \to E{+}E, E \to E^{\star}, E \to (E)\}$$

E -> $\boldsymbol{\varepsilon}$, E -> $\boldsymbol{\phi}$

Example: (simple) expressions in a typical prog lang. Operators are $+$ and *, and arguments are identfiers, i.e. strings in $L((\boldsymbol{a}+\boldsymbol{b})(\boldsymbol{a}+\boldsymbol{b}+\mathbf{0}+\mathbf{1})^*)$ e.g, ***a*(a + b00)***

The expressions are defined by the grammar

$$G = (\{E, I\}, T, P, E)$$

where $T = \{+, *, (,), a, b, 0, 1\}$ and P is the following set of productions:

1. $E \rightarrow I$
2. $E \rightarrow E + E$
3. $E \rightarrow E * E$
4. $E \rightarrow (E)$
5. $I \rightarrow a$
6. $I \rightarrow b$
7. $I \rightarrow Ia$
8. $I \rightarrow Ib$
9. $I \rightarrow I0$
10. $I \rightarrow I1$

Derivations using grammars

- *Recursive inference*, using productions from body to head

- *Derivations*, using productions from head to body.

Example of recursive inference:

	String	Lang	Prod	String(s) used
(i)	a	I	5	-
(ii)	b	I	6	-
(iii)	$b0$	I	9	*(ii)*
(iv)	$b00$	I	9	*(iii)*
(v)	a	E	1	*(i)*
(vi)	$b00$	E	1	*(iv)*
(vii)	$a + b00$	E	2	*(v)*, *(vi)*
(viii)	$(a + b00)$	E	4	*(vii)*
(ix)	$a * (a + b00)$	E	3	*(v)*, *(viii)*

Let $G = (V, T, P, S)$ be a CFG, $A \in V$, $\{\alpha, \beta\} \subset (V \cup T)^*$, and $A \to \gamma \in P$.

Then we write

$$\alpha A \beta \underset{G}{\Rightarrow} \alpha \gamma \beta$$

or, if G is understood

$$\alpha A \beta \Rightarrow \alpha \gamma \beta$$

and say that $\alpha A \beta$ *derives* $\alpha \gamma \beta$.

We define $\overset{*}{\Rightarrow}$ to be the reflexive and transitive closure of $\Rightarrow$, IOW:

Basis: Let $\alpha \in (V \cup T)^*$. Then $\alpha \overset{*}{\Rightarrow} \alpha$.

Induction: If $\alpha \overset{*}{\Rightarrow} \beta$, and $\beta \Rightarrow \gamma$, then $\alpha \overset{*}{\Rightarrow} \gamma$.

Example: Derivation of $a * (a + b00)$ from E in the grammar of slide 138:

$$E \Rightarrow E * E \Rightarrow I * E \Rightarrow a * E \Rightarrow a * (E) \Rightarrow$$

$$a*(E+E) \Rightarrow a*(I+E) \Rightarrow a*(a+E) \Rightarrow a*(a+I) \Rightarrow$$

$$a * (a + I0) \Rightarrow a * (a + I00) \Rightarrow a * (a + b00)$$

So, we can write $E \stackrel{*}{\Rightarrow} a * (a + b00)$.

Note: At each step we might have several rules to choose from, e.g.
$I * E \Rightarrow a * E \Rightarrow a * (E)$, versus
$I * E \Rightarrow I * (E) \Rightarrow a * (E)$.

Note2: Not all choices lead to successful derivations of a particular string, for instance

$$E \Rightarrow E + E$$

won't lead to a derivation of $a * (a + b00)$.

Leftmost and Rightmost Derivations

Leftmost derivation $\underset{lm}{\Rightarrow}$: Always replace the left-most variable by one of its rule-bodies.

Rightmost derivation $\underset{rm}{\Rightarrow}$: Always replace the rightmost variable by one of its rule-bodies.

Leftmost: The derivation on the previous slide.

Rightmost:

$$E \underset{rm}{\Rightarrow} E * E \underset{rm}{\Rightarrow}$$

$$E*(E) \underset{rm}{\Rightarrow} E*(E+E) \underset{rm}{\Rightarrow} E*(E+I) \underset{rm}{\Rightarrow} E*(E+I0)$$

$$\underset{rm}{\Rightarrow} E*(E+I00) \underset{rm}{\Rightarrow} E*(E+b00) \underset{rm}{\Rightarrow} E*(I+b00)$$

$$\underset{rm}{\Rightarrow} E*(a+b00) \underset{rm}{\Rightarrow} I*(a+b00) \underset{rm}{\Rightarrow} a*(a+b00)$$

We can conclude that $E \overset{*}{\underset{rm}{\Rightarrow}} a*(a+b00)$

The Language of a Grammar

If $G(V, T, P, S)$ is a CFG, then the *language of* G is

$$L(G) = \{w \in T^* : S \overset{*}{\underset{G}{\Rightarrow}} w\}$$

i.e. the set of strings over T^* derivable from the start symbol.

If G is a CFG, we call $L(G)$ a *context-free language (or CFL)*.

Example: $L(G_{pal})$ is a context-free language.

Theorem 5.7:

$$L(G_{pal}) = \{w \in \{0,1\}^* : w = w^R\}$$

Proof: ($\supseteq$-direction.) Suppose $w = w^R$. We show by induction on $|w|$ that $w \in L(G_{pal})$

Basis: $|w| = 0$, or $|w| = 1$. Then w is $\epsilon, 0$, or 1. Since $P \to \epsilon, P \to 0$, and $P \to 1$ are productions, we conclude that $P \overset{*}{\underset{G}{\Rightarrow}} w$ in all base cases.

Induction: Suppose $|w| \geq 2$. Since $w = w^R$, we have $w = 0x0$, or $w = 1x1$, and $x = x^R$.

If $w = 0x0$ we know from the IH that $P \overset{*}{\Rightarrow} x$. Then

$$P \Rightarrow 0P0 \overset{*}{\Rightarrow} 0x0 = w$$

Thus $w \in L(G_{pal})$.

The case for $w = 1x1$ is similar.

($\subseteq$-direction.) We assume that $w \in L(G_{pal})$ and must show that $w = w^R$.

Since $w \in L(G_{pal})$, we have $P \stackrel{*}{\Rightarrow} w$.

We do an induction on the length of $\stackrel{*}{\Rightarrow}$.

Basis: The derivation $P \stackrel{*}{\Rightarrow} w$ is done in one step.

Then w must be $\epsilon, 0$, or 1, all palindromes.

Induction: Let $n \geq 1$, and suppose the derivation takes $n+1$ steps. Then we must have

$$w = 0x0 \stackrel{*}{\Leftarrow} 0P0 \Leftarrow P$$

Hence, $P \stackrel{*}{\Rightarrow} x$

or

$$w = 1x1 \stackrel{*}{\Leftarrow} 1P1 \Leftarrow P$$

where the second derivation is done in n steps.

By the IH x is a palindrome, and the inductive proof is complete.

Ex. Design CFGs for the following languages:
L_1 = {balanced parentheses} = {ε,(),(()),()(),((())),()(()),(())(), ...} (the Dyck language)
$L_2 = \{0^m 1^n 2^p \mid m,n, p >= 0, m = n + p\}$
$L_3 = \{w \mid w \in \{0,1\}^*, w <> w^R\}$

Sentential Forms

Let $G = (V, T, P, S)$ be a CFG, and $\alpha \in (V \cup T)^*$. If

$$S \overset{*}{\Rightarrow} \alpha$$

we say that α is a *sentential form*.

If $S \underset{lm}{\Rightarrow} \alpha$ we say that α is a *left-sentential form*, and if $S \underset{rm}{\Rightarrow} \alpha$ we say that α is a *right-sentential form*

Note: $L(G)$ is those sentential forms that are in T^*. (i.e., sentences)

Example: Take G from slide 138. Then $E * (I + E)$ is a sentential form since

$$E \Rightarrow E*E \Rightarrow E*(E) \Rightarrow E*(E+E) \Rightarrow E*(I+E)$$

This derivation is neither leftmost, nor rightmost

Example: $a * E$ is a left-sentential form, since

$$E \underset{lm}{\Rightarrow} E * E \underset{lm}{\Rightarrow} I * E \underset{lm}{\Rightarrow} a * E$$

Example: $E*(E+E)$ is a right-sentential form, since

$$E \underset{rm}{\Rightarrow} E * E \underset{rm}{\Rightarrow} E * (E) \underset{rm}{\Rightarrow} E * (E + E)$$

Parse Trees

- If $w \in L(G)$, for some CFG, then w has a *parse tree*, which tells us the (syntactic) structure of w

- w could be a program, a SQL-query, an XML-document, etc.

- Parse trees are an alternative representation to derivations and recursive inferences.

- There can be several parse trees for the same string

- Ideally there should be only one parse tree (the "true" structure) for each string, i.e. the language should be *unambiguous*.

- Unfortunately, we cannot always remove the ambiguity.

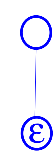
Constructing Parse Trees

Let $G = (V, T, P, S)$ be a CFG. A tree is a *parse tree* for G if:

rooted, ordered

1. Each interior node is labelled by a variable in V.

2. Each leaf is labelled by a symbol in $V \cup T \cup \{\epsilon\}$. Any ϵ-labelled leaf is the only child of its parent.



3. If an interior node is lablelled A, and its children (from left to right) labelled

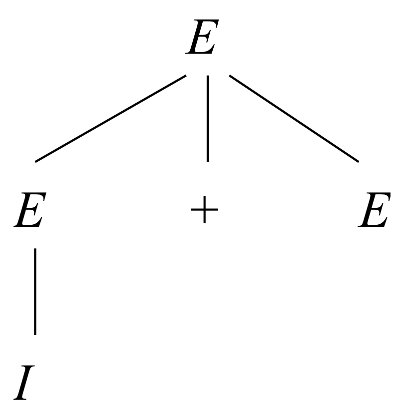
$$X_1, X_2, \ldots, X_k,$$

then $A \to X_1 X_2 \ldots X_k \in P$.

Example: In the grammar

$$
\begin{aligned}
&1.\ E \rightarrow I \\
&2.\ E \rightarrow E + E \\
&3.\ E \rightarrow E * E \\
&4.\ E \rightarrow (E) \\
&\quad\quad\vdots
\end{aligned}
$$

the following is a parse tree:

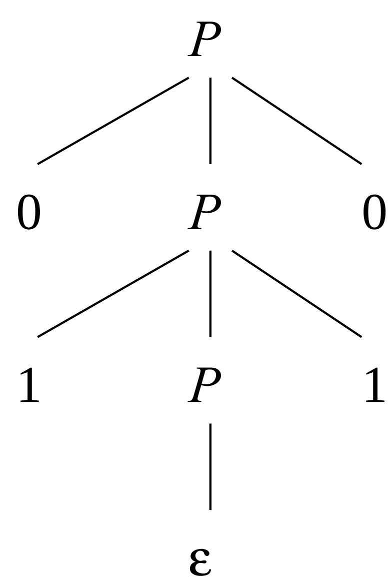


This parse tree shows the derivation $E \stackrel{*}{\Rightarrow} I + E$

Example: In the grammar

1. $P \rightarrow \epsilon$
2. $P \rightarrow 0$
3. $P \rightarrow 1$
4. $P \rightarrow 0P0$
5. $P \rightarrow 1P1$

the following is a parse tree:



It shows the derivation of $P \stackrel{*}{\Rightarrow} 0110$.

The Yield of a Parse Tree

The *yield* of a parse tree is the string of leaves from left to right.

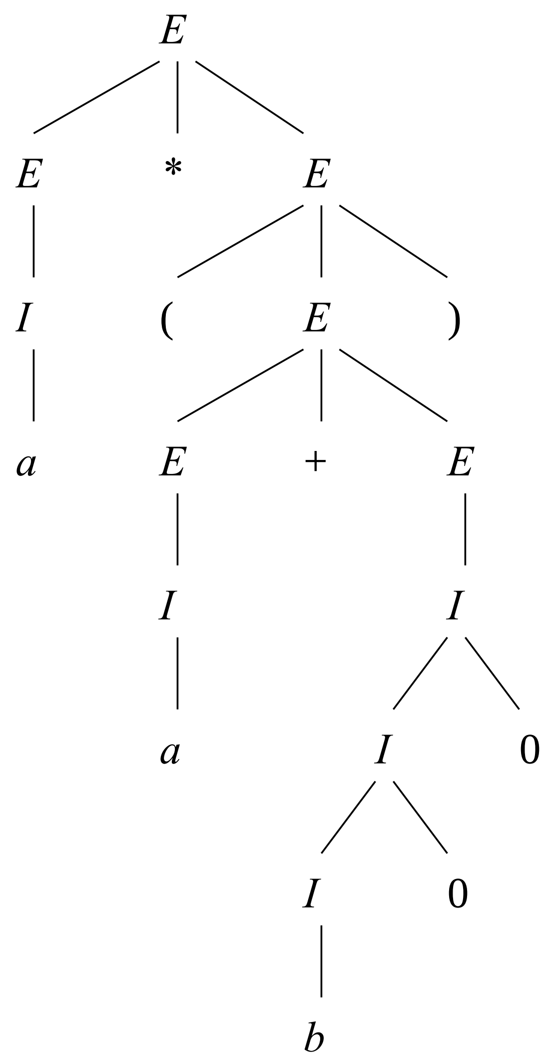
Important are those parse trees where:

1. The yield is a terminal string.

2. The root is labelled by the start symbol

We shall see the the set of yields of these important parse trees is the language of the grammar.

Example: Below is an important parse tree



The yield is $a * (a + b00)$.

Compare the parse tree with the derivation on slide 141.

Let $G = (V, T, P, S)$ be a CFG, and $A \in V$. We are going to show that the following are equivalent:

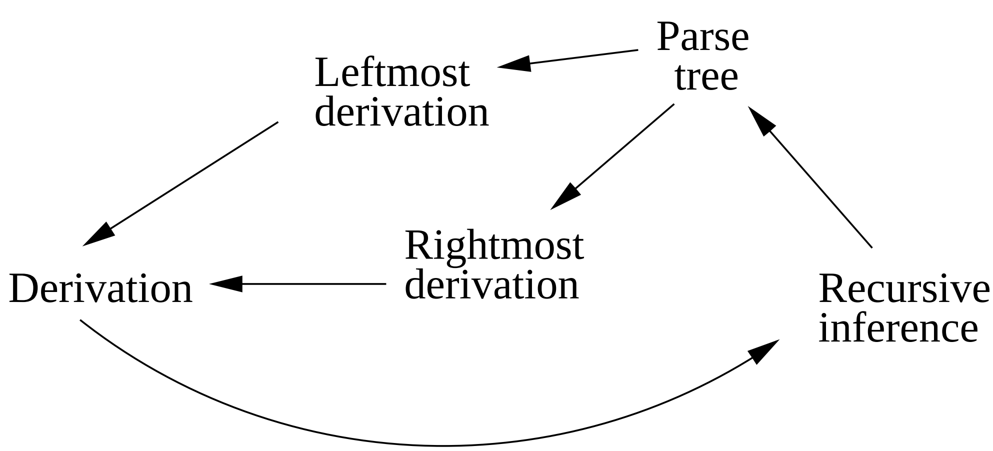
1. We can determine by recursive inference that w is in the language of A

2. $A \stackrel{*}{\Rightarrow} w$

3. $A \underset{lm}{\stackrel{*}{\Rightarrow}} w$, and $A \underset{rm}{\stackrel{*}{\Rightarrow}} w$

4. There is a parse tree of G with root A and yield w.

To prove the equivalences, we use the following plan.



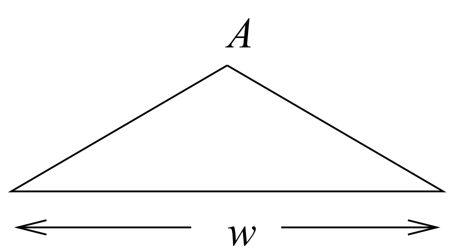
From Inferences to Trees

Theorem 5.12: Let $G = (V, T, P, S)$ be a CFG, and suppose we can show w to be in the language of a variable A. Then there is a parse tree for G with root A and yield w.

by inference

Proof: We do an induction of the length of the inference.

Basis: One step. Then we must have used a production $A \rightarrow w$. The desired parse tree is then



Induction: w is inferred in $n+1$ steps. Suppose the last step was based on a production

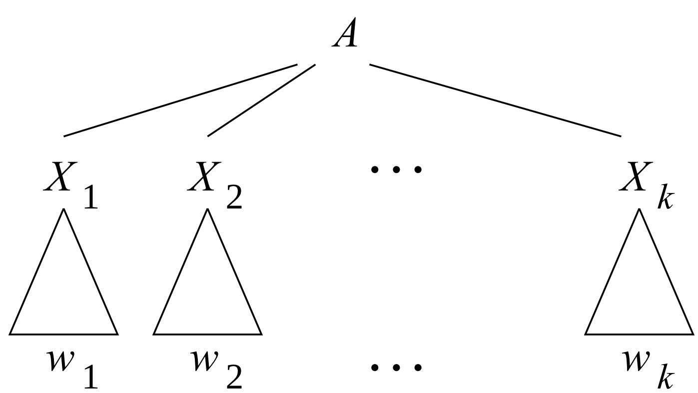
$$A \to X_1 X_2 \cdots X_k,$$

where $X_i \in V \cup T$. We break w up as

$$w_1 w_2 \cdots w_k,$$

where $w_i = X_i$, when $X_i \in T$, and when $X_i \in V$, then w_i was previously inferred being in $L(X_i)$, in at most n steps.

By the IH there are parse trees i with root X_i and yield w_i. Then the following is a parse tree for G with root A and yield w:



From trees to derivations

We'll show how to construct a leftmost derivation from a parse tree.

Example: In the grammar of slide 138 there clearly is a derivation

$$E \Rightarrow I \Rightarrow Ib \Rightarrow ab.$$

Then, for any α and β there is a derivation

$$\alpha E\beta \Rightarrow \alpha I\beta \Rightarrow \alpha Ib\beta \Rightarrow \alpha ab\beta.$$

For example, suppose we have a derivation

$$E \Rightarrow E + E \Rightarrow E + (E).$$

Then we can choose $\alpha = E+($ and $\beta =)$ and continue the derivation as

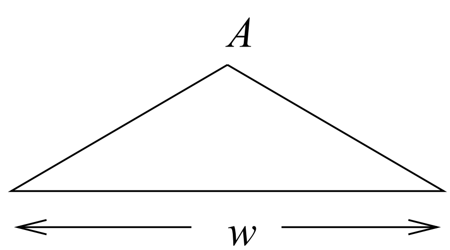
$$E + (E) \Rightarrow E + (I) \Rightarrow E + (Ib) \Rightarrow E + (ab).$$

This is why CFG's are called context-free.

Theorem 5.14: Let $G = (V, T, P, S)$ be a CFG, and suppose there is a parse tree with root labelled A and yield w. Then $A \underset{lm}{\overset{*}{\Rightarrow}} w$ in G.

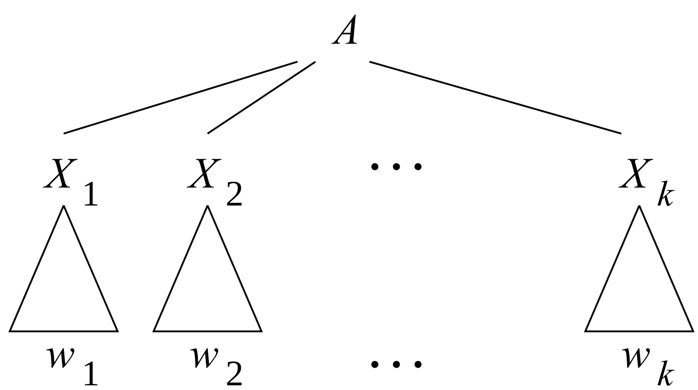
Proof: We do an induction on the height of the parse tree.

Basis: Height is 1. The tree must look like



Consequently $A \to w \in P$, and $A \underset{lm}{\Rightarrow} w$.

Induction: Height is $n + 1$. The tree must look like



Then $w = w_1 w_2 \cdots w_k$, where

1. If $X_i \in T$, then $w_i = X_i$.

2. If $X_i \in V$, then $X_i \underset{lm}{\overset{*}{\Rightarrow}} w_i$ in G by the IH.

Now we construct $A \underset{lm}{\overset{*}{\Rightarrow}} w$ by an (inner) induction by showing that

$$\forall i : A \underset{lm}{\overset{*}{\Rightarrow}} w_1 w_2 \cdots w_i X_{i+1} X_{i+2} \cdots X_k.$$

Basis: Let $i = 0$. We already know that
$A \underset{lm}{\Rightarrow} X_1 X_{i+2} \cdots X_k$.

Induction: Make the IH that

$$A \underset{lm}{\overset{*}{\Rightarrow}} w_1 w_2 \cdots w_{i-1} X_i X_{i+1} \cdots X_k.$$

(*Case 1:*) $X_i \in T$. Do nothing, since $X_i = w_i$ gives us

$$A \underset{lm}{\overset{*}{\Rightarrow}} w_1 w_2 \cdots w_i X_{i+1} \cdots X_k.$$

(*Case 2:*) $X_i \in V$. By the IH there is a derivation $X_i \underset{lm}{\Rightarrow} \alpha_1 \underset{lm}{\Rightarrow} \alpha_2 \underset{lm}{\Rightarrow} \cdots \underset{lm}{\Rightarrow} w_i$. By the contex-free property of derivations we can proceed with

$$A \overset{*}{\underset{lm}{\Rightarrow}}$$

$$w_1 w_2 \cdots w_{i-1} X_i X_{i+1} \cdots X_k \underset{lm}{\Rightarrow}$$

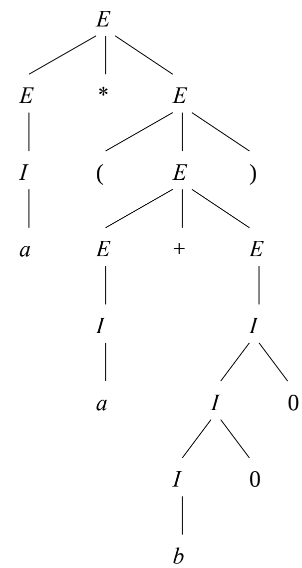
$$w_1 w_2 \cdots w_{i-1} \alpha_1 X_{i+1} \cdots X_k \underset{lm}{\Rightarrow}$$

$$w_1 w_2 \cdots w_{i-1} \alpha_2 X_{i+1} \cdots X_k \underset{lm}{\Rightarrow}$$

$$\cdots$$

$$w_1 w_2 \cdots w_{i-1} w_i X_{i+1} \cdots X_k$$

Example: Let's construct the leftmost derivation for the tree



Suppose we have inductively constructed the leftmost derivation

$$E \underset{lm}{\Rightarrow} I \underset{lm}{\Rightarrow} a$$

corresponding to the leftmost subtree, and the leftmost derivation

$$E \underset{lm}{\Rightarrow} (E) \underset{lm}{\Rightarrow} (E + E) \underset{lm}{\Rightarrow} (I + E) \underset{lm}{\Rightarrow} (a + E) \underset{lm}{\Rightarrow}$$
$$(a + I) \underset{lm}{\Rightarrow} (a + I0) \underset{lm}{\Rightarrow} (a + I00) \underset{lm}{\Rightarrow} (a + b00)$$

corresponding to the righmost subtree.

For the derivation corresponding to the whole tree we start with $E \underset{lm}{\Rightarrow} E * E$ and expand the first E with the first derivation and the second E with the second derivation:

$$
\begin{aligned}
& E \underset{lm}{\Rightarrow} \\
& E * E \underset{lm}{\Rightarrow} \\
& I * E \underset{lm}{\Rightarrow} \\
& a * E \underset{lm}{\Rightarrow} \\
& a * (E) \underset{lm}{\Rightarrow} \\
& a * (E + E) \underset{lm}{\Rightarrow} \\
& a * (I + E) \underset{lm}{\Rightarrow} \\
& a * (a + E) \underset{lm}{\Rightarrow} \\
& a * (a + I) \underset{lm}{\Rightarrow} \\
& a * (a + I0) \underset{lm}{\Rightarrow} \\
& a * (a + I00) \underset{lm}{\Rightarrow} \\
& a * (a + b00)
\end{aligned}
$$

From Derivations to Recursive Inferences

Observation: Suppose that $A \Rightarrow X_1 X_2 \cdots X_k \stackrel{*}{\Rightarrow} w$. Then $w = w_1 w_2 \cdots w_k$, where $X_i \stackrel{*}{\Rightarrow} w_i$

The factor w_i can be extracted from $A \stackrel{*}{\Rightarrow} w$ by looking at the expansion of X_i only.

Example: $E \stackrel{*}{\Rightarrow} a * b + a$, and

$$E \Rightarrow \underbrace{E}_{X_1} \underbrace{*}_{X_2} \underbrace{E}_{X_3} \underbrace{+}_{X_4} \underbrace{E}_{X_5}$$

We have

$$E \Rightarrow E * E + E \Rightarrow I * E + E \Rightarrow I * I + E \Rightarrow$$
$$I * I + I \Rightarrow a * I + I \Rightarrow a * b + I \Rightarrow a * b + a$$

By looking at the expansion of $X_3 = E$ only, we can extract

$$E \Rightarrow I \Rightarrow b.$$

Theorem 5.18: Let $G = (V, T, P, S)$ be a CFG. Suppose $A \underset{G}{\overset{*}{\Rightarrow}} w$, and that w is a string of terminals. Then we can infer that w is in the language of variable A.

Proof: We do an induction on the length of the derivation $A \underset{G}{\overset{*}{\Rightarrow}} w$.

Basis: One step. If $A \underset{G}{\Rightarrow} w$ there must be a production $A \rightarrow w$ in P. The we can infer that w is in the language of A.

Induction: Suppose $A \underset{G}{\overset{*}{\Rightarrow}} w$ in $n+1$ steps. Write the derivation as

$$A \underset{G}{\Rightarrow} X_1 X_2 \cdots X_k \underset{G}{\overset{*}{\Rightarrow}} w$$

The as noted on the previous slide we can break w as $w_1 w_2 \cdots w_k$ where $X_i \underset{G}{\overset{*}{\Rightarrow}} w_i$. Furthermore, $X_i \underset{G}{\overset{*}{\Rightarrow}} w_i$ can use at most n steps.

Now we have a production $A \to X_1 X_2 \cdots X_k$, and we know by the IH that we can infer w_i to be in the language of X_i.

Therefore we can infer $w_1 w_2 \cdots w_k$ to be in the language of A.

Gram Matic (Paul Cernea):
https://itunes.apple.com/ca/app/gram-matic/id914302373?mt=8

Ambiguity in Grammars and Languages

In the grammar

$$
\begin{array}{ll}
1. & E \rightarrow I \\
2. & E \rightarrow E + E \\
3. & E \rightarrow E * E \\
4. & E \rightarrow (E) \\
 & \cdots
\end{array}
$$

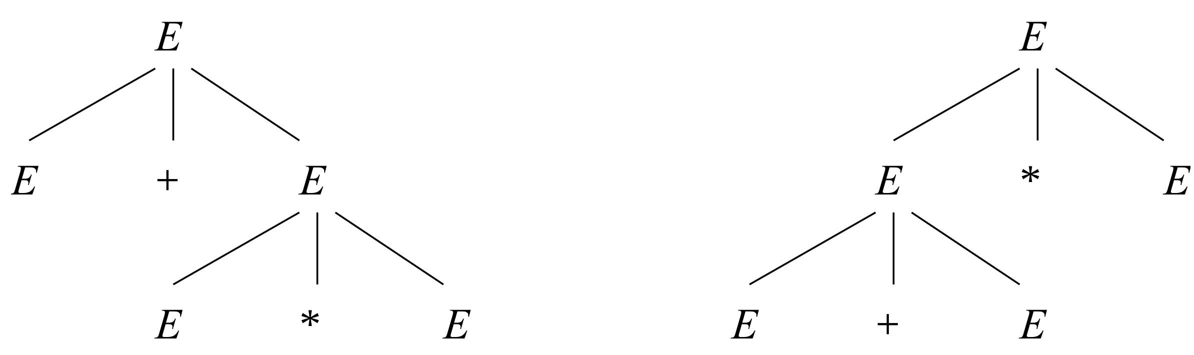
the sentential form $E + E * E$ has two derivations:

$$E \Rightarrow E + E \Rightarrow E + E * E$$

and

$$E \Rightarrow E * E \Rightarrow E + E * E$$

This gives us two parse trees:



(a) (b)

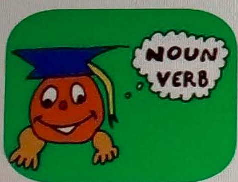
https://itunes.apple.com/us/app/gram-matic/id914302373?mt=8

View More by This Developer

Gram Matic

By Paul Cernea

Open iTunes to buy and download apps.



View in iTunes

$0.99

Category: Education
Released: Nov 18, 2014
Version: 1.0
Size: 12.8 MB
Language: English
Seller: Paul Cernea
© Paul Cernea
Rated 4+

Compatibility: Requires iOS 7.1 or later. Compatible with iPhone, iPad, and iPod touch.

Customer Ratings

We have not received enough ratings to display an average for the current version of this application.

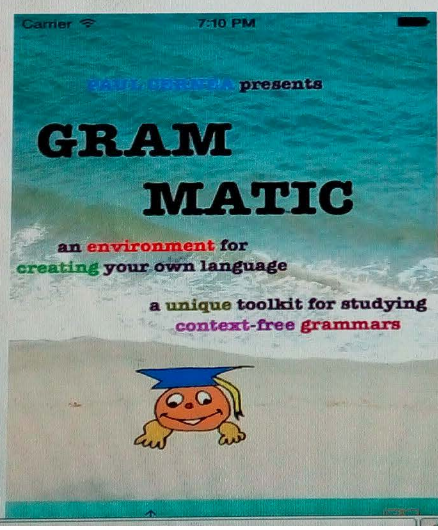
More iPhone Apps by Paul Cernea

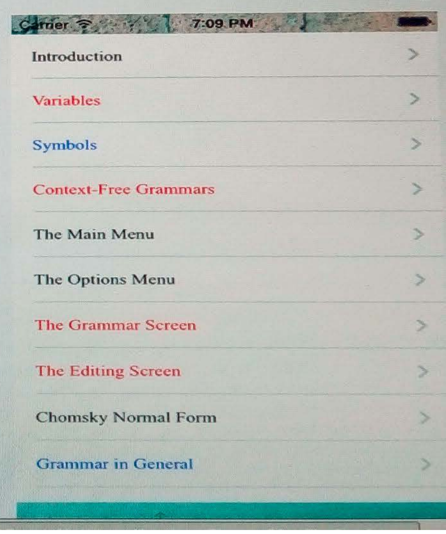
Description

Have you dreamt of inventing your own language and seeing it come alive in real time? Are you a teacher who wants to make grammar interactive and colorful for your students? Are you a computer scientist or linguist who wants to play around with grammars and parsing to gain intuition? This is the app for you. Design your own context-free grammar on the go. See it converted to Chomsky normal form before your eyes. Type in text and immediately see if it belongs to the language generated by your grammar. An immersive creative and educational tool in the palm of your hands! Bonus: email grammars to friends and convert them to PDF!

Gram Matic Support ›

iPhone Screenshots





The mere existence of several *derivations* is not dangerous, it is the existence of several parse trees that ruins a grammar.

But, multiple left-most (or right-most) derivations do cause ambiguity.

Example: In the same grammar

$$
\begin{aligned}
5.\quad & I \rightarrow a \\
6.\quad & I \rightarrow b \\
7.\quad & I \rightarrow Ia \\
8.\quad & I \rightarrow Ib \\
9.\quad & I \rightarrow I0 \\
10.\quad & I \rightarrow I1
\end{aligned}
$$

the string $a + b$ has several derivations, e.g.

$$E \Rightarrow E + E \Rightarrow I + E \Rightarrow a + E \Rightarrow a + I \Rightarrow a + b$$

and

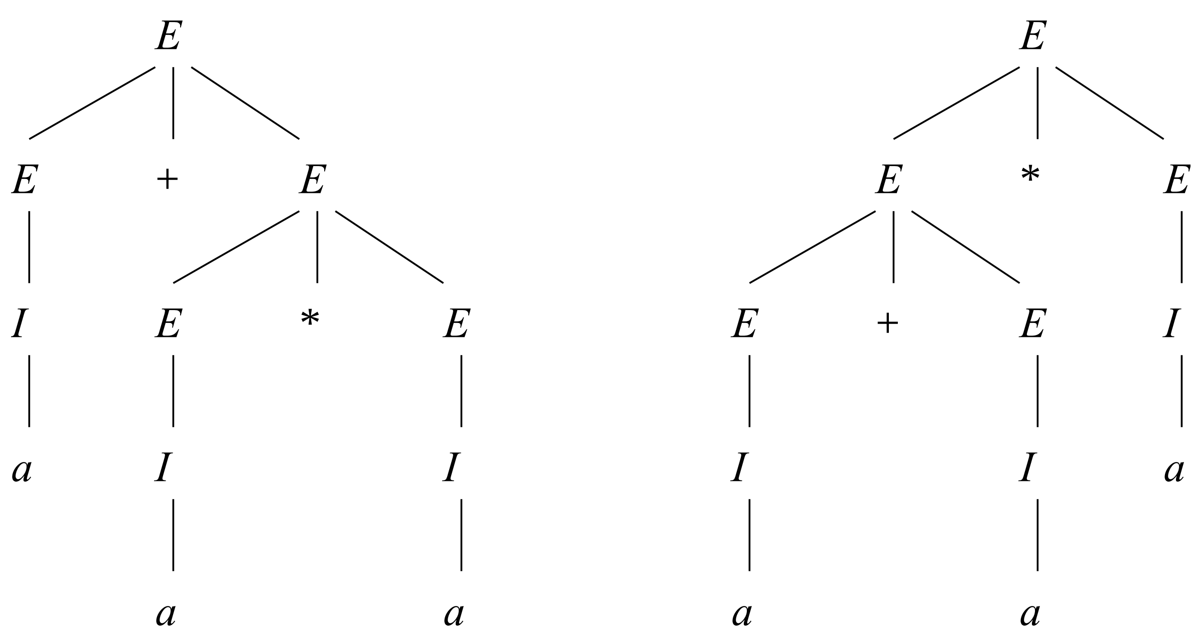
$$E \Rightarrow E + E \Rightarrow E + I \Rightarrow I + I \Rightarrow I + b \Rightarrow a + b$$

However, their parse trees are the same, and the structure of $a + b$ is unambiguous.

Definition: Let $G = (V, T, P, S)$ be a CFG. We say that G is *ambiguous* if there is a string in T^* that has more than one parse tree rooted at S.

If every string in $L(G)$ has at most one parse tree, G is said to be *unambiguous*.

Example: The terminal string $a + a * a$ has two parse trees:



(a) (b)

Example: Unambiguous Grammar

B -> (RB | ϵ   R ->) | (RR

- Construct a unique leftmost derivation for a given balanced string of parentheses by scanning the string from left to right.
  - If we need to expand B, then use B -> (RB if the next symbol is "(" and ϵ if at the end.
  - If we need to expand R, use R ->) if the next symbol is ")" and (RR if it is "(".

The Parsing Process

Remaining Input:

(())()

Next symbol

Steps of leftmost derivation:

B

B -> (RB | ϵ   R ->) | (RR

The Parsing Process

Remaining Input:

())()

Next symbol

Steps of leftmost derivation:

B

(RB

B -> (RB | ϵ   R ->) | (RR

The Parsing Process

Remaining Input:

))()

Next symbol

Steps of leftmost derivation:

B

(RB

((RRB

B -> (RB | ϵ   R ->) | (RR

The Parsing Process

Remaining Input:

)()

Next symbol

Steps of leftmost derivation:

B

(RB

((RRB

(()RB

B -> (RB | ϵ R ->) | (RR

The Parsing Process

Remaining Input:

()

Next
symbol

Steps of leftmost derivation:

B

(RB

((RRB

(()RB

(())B

B -> (RB | ϵ   R ->) | (RR

The Parsing Process

Remaining Input:

)

Next
symbol

Steps of leftmost derivation:

B

(RB

((RRB

(()RB

(())B

(())(RB

B -> (RB | ϵ R ->) | (RR

The Parsing Process

Remaining Input:

Next
symbol

Steps of leftmost derivation:

B

(RB

((RRB

(()RB

(())B

(())(RB

(())()B

B -> (RB | ε   R ->) | (RR

The Parsing Process

Remaining Input:

Next
symbol

Steps of leftmost derivation:

B

(RB

((RRB

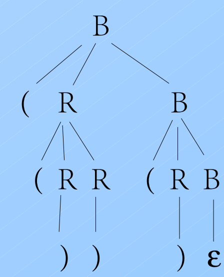
(()RB

(())B

(())(RB

(())()B

(())()



B -> (RB | ϵ R ->) | (RR

LL(1) Grammars

- As an aside, a grammar like B -> (RB | ϵ R ->) | (RR, where you can always figure out the production to use in a leftmost derivation by scanning the given string left-to-right and looking only at the next one symbol is called LL(1).
  - "Leftmost derivation, left-to-right scan, one symbol of lookahead."

LL(1) Grammars – (2)

- Most programming languages have LL(1) grammars.
- LL(1) grammars are never ambiguous.

Ex. Prove the CFG for Dyck language B -> (B)B | ε

is LL(1).

Removing Ambiguity From Grammars

Good news: Sometimes we can remove ambiguity "by hand" (without changing the language)

Bad news: There is no algorithm to do it

More bad news: Some CFL's have only ambiguous CFG's

We are studying the grammar

$$E \rightarrow I \mid E + E \mid E * E \mid (E)$$
$$I \rightarrow a \mid b \mid Ia \mid Ib \mid I0 \mid I1$$

There are two problems:

1. There is no precedence between * and +

2. There is no grouping of sequences of operators, e.g. is $E + E + E$ meant to be $E + (E + E)$ or $(E + E) + E$.

Solution: We introduce more variables, each representing expressions of same "binding strength".

1. A *factor* is an expresson that cannot be broken apart by an adjacent * or $+$. Our factors are

   (a) Identifiers

   (b) A parenthesized expression.

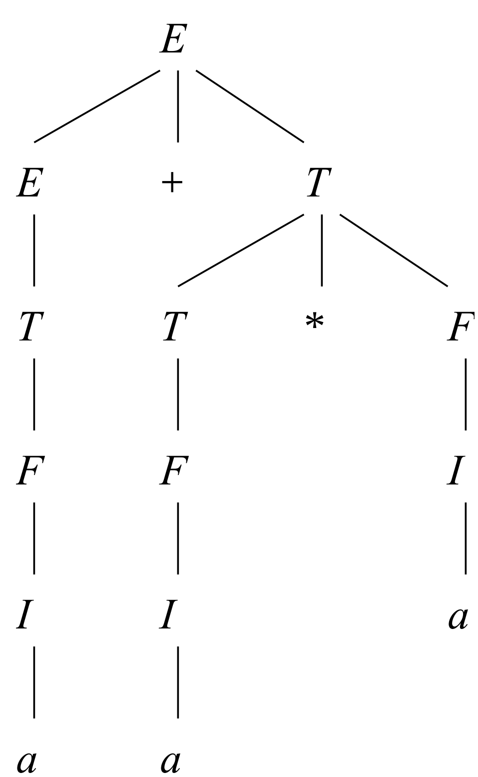
2. A *term* is an expresson that cannot be broken by $+$. For instance $a * b$ can be broken by $a1*$ or $*a1$. It cannot be broken by $+$, since e.g. $a1 + a * b$ is (by precedence rules) same as $a1 + (a * b)$, and $a * b + a1$ is same as $(a * b) + a1$.

3. The rest are *expressions*, i.e. they can be broken apart with * or $+$.

We'll let F stand for factors, T for terms, and E for expressions. Consider the following grammar:

1. $I \rightarrow a \mid b \mid Ia \mid Ib \mid I0 \mid I1$
2. $F \rightarrow I \mid (E)$
3. $T \rightarrow F \mid T * F$
4. $E \rightarrow T \mid E + T$

Now the only parse tree for $a + a * a$ will be



Why is the new grammar unambiguous?

Intuitive explanation:

- A factor is either an identifier or (E), for some expression E.

- The only parse tree for a sequence

$$f_1 * f_2 * \cdots * f_{n-1} * f_n$$

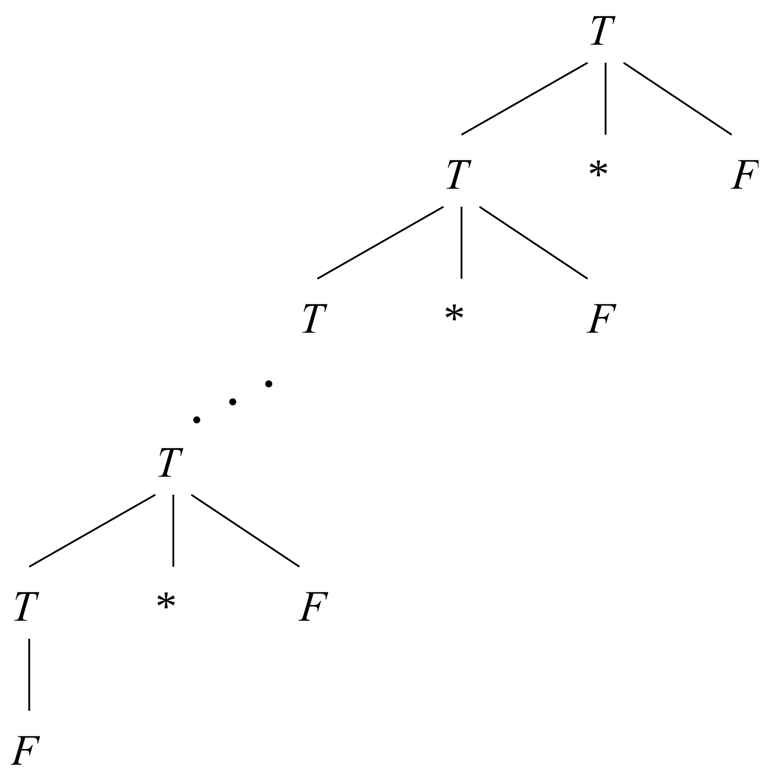
of factors is the one that gives $f_1 * f_2 * \cdots * f_{n-1}$ as a term and f_n as a factor, as in the parse tree on the next slide.

IOW, consecutive multiplications are calculated from left to right.

- An expression is a sequence

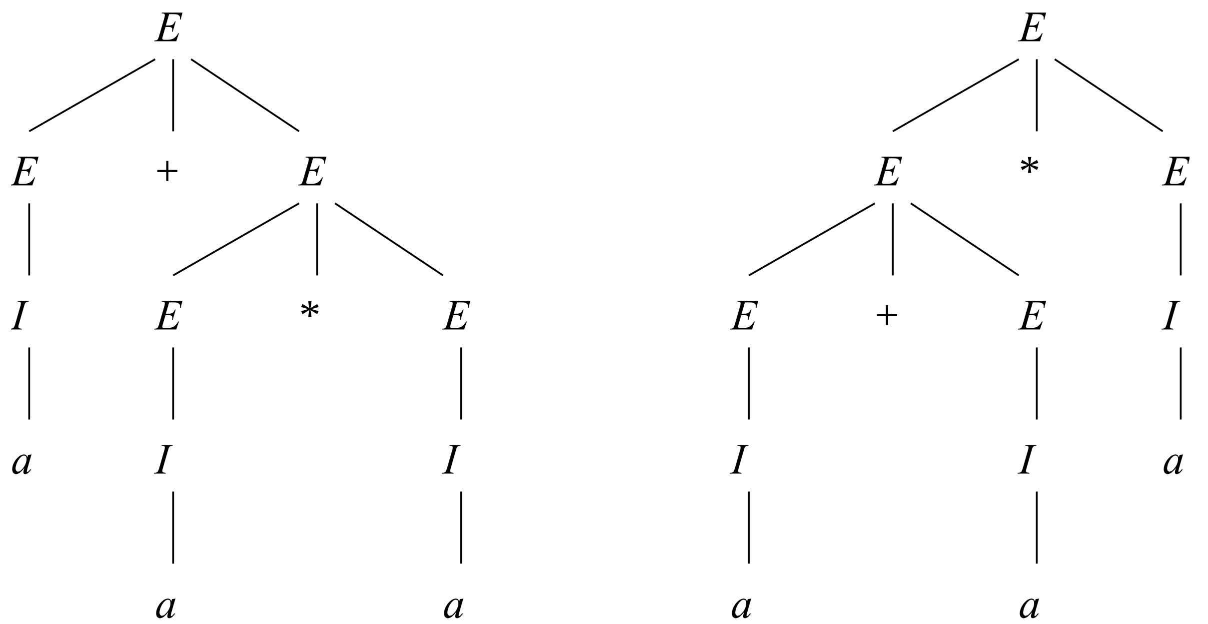
$$t_1 + t_2 + \cdots + t_{n-1} + t_n$$

of terms t_i. It can only be parsed with $t_1 + t_2 + \cdots + t_{n-1}$ as an expression and t_n as a term.


T
T
*
F
T
*
F
T
T
*
F
F

Leftmost derivations and Ambiguity

The two parse trees for $a + a * a$



(a) (b)

give rise to two derivations:

$$E \underset{lm}{\Rightarrow} E + E \underset{lm}{\Rightarrow} I + E \underset{lm}{\Rightarrow} a + E \underset{lm}{\Rightarrow} a + E * E$$

$$\underset{lm}{\Rightarrow} a + I * E \underset{lm}{\Rightarrow} a + a * E \underset{lm}{\Rightarrow} a + a * I \underset{lm}{\Rightarrow} a + a * a$$

and

$$E \underset{lm}{\Rightarrow} E * E \underset{lm}{\Rightarrow} E + E * E \underset{lm}{\Rightarrow} I + E * E \underset{lm}{\Rightarrow} a + E * E$$

$$\underset{lm}{\Rightarrow} a + I * E \underset{lm}{\Rightarrow} a + a * E \underset{lm}{\Rightarrow} a + a * I \underset{lm}{\Rightarrow} a + a * a$$

In General:

- One parse tree, but many derivations

- Many *leftmost* derivation implies many parse trees.

- Many *rightmost* derivation implies many parse trees.

Theorem 5.29: For any CFG G, a terminal string w has two distinct parse trees if and only if w has two distinct leftmost derivations from the start symbol.

Sketch of Proof: *(Only If.)* If the two parse trees differ, they have a node with different productions, say $A \rightarrow X_1 X_2 \cdots X_k$ and $A \rightarrow Y_1 Y_2 \cdots Y_m$. The corresponding leftmost derivations will use derivations based on these two different productions and will thus be distinct.

(If.) Let's look at how we construct a parse tree from a leftmost derivation. It should now be clear that two distinct derivations gives rise to two different parse trees.

Inherent Ambiguity

A CFL L is *inherently ambiguous* if *all* grammars for L are ambiguous.

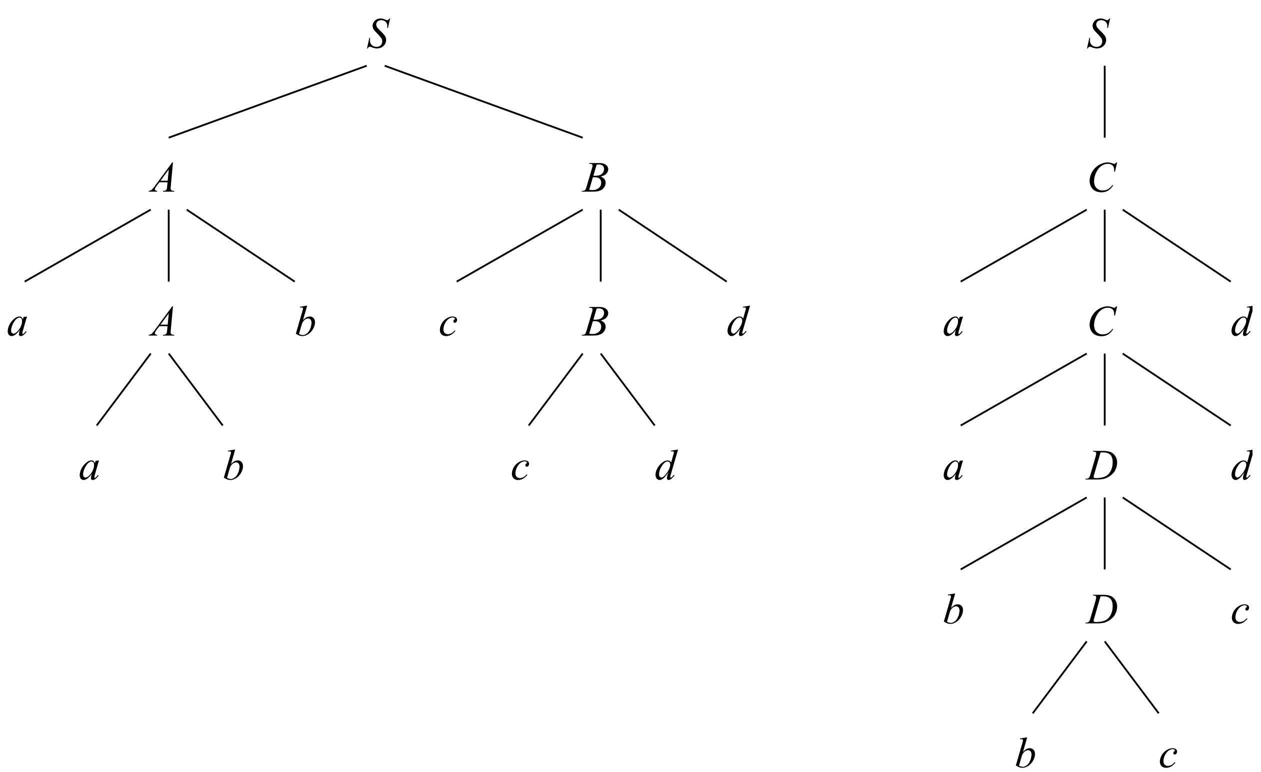
Example: Consider $L =$

$\{a^n b^n c^m d^m : n \geq 1, m \geq 1\} \cup \{a^n b^m c^m d^n : n \geq 1, m \geq 1\}$.

A grammar for L is

$$
\begin{aligned}
S &\rightarrow AB \mid C \\
A &\rightarrow aAb \mid ab \\
B &\rightarrow cBd \mid cd \\
C &\rightarrow aCd \mid aDd \\
D &\rightarrow bDc \mid bc
\end{aligned}
$$

Let's look at parsing the string $aabbccdd$.



(a) (b)

From this we see that there are two leftmost derivations:

$$S \underset{lm}{\Rightarrow} AB \underset{lm}{\Rightarrow} aAbB \underset{lm}{\Rightarrow} aabbB \underset{lm}{\Rightarrow} aabbcBd \underset{lm}{\Rightarrow} aabbccdd$$

and

$$S \underset{lm}{\Rightarrow} C \underset{lm}{\Rightarrow} aCd \underset{lm}{\Rightarrow} aaDdd \underset{lm}{\Rightarrow} aabDcdd \underset{lm}{\Rightarrow} aabbccdd$$

It can be shown that *every* grammar for L behaves like the one above. The language L is inherently ambiguous.

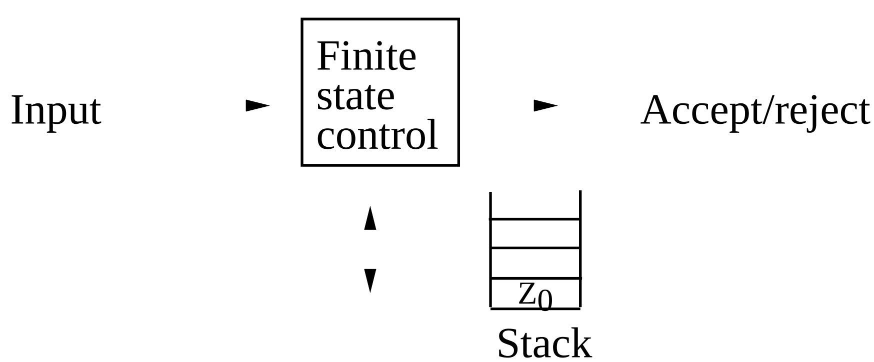
There is no algorithm to determine if a CFL is inherently ambiguous.
There is no algorithm to determine if a CFG is ambiguous.

Pushdown Automata

A pushdown automaton (PDA) is essentially an ϵ-NFA with a stack.

On a transition the PDA:

1. Consumes an input symbol. $\boxed{\text{or } \varepsilon}$
2. Goes to a new state (or stays in the old).
3. Replaces the top of the stack by any string (does nothing, pops the stack, or pushes a string onto the stack)



Example: Let's consider

$$L_{wwr} = \{ww^R : w \in \{0,1\}^*\},$$

with "grammar" $P \rightarrow 0P0$, $P \rightarrow 1P1$, $P \rightarrow \epsilon$. A PDA for L_{wwr} has three states, and operates as follows:

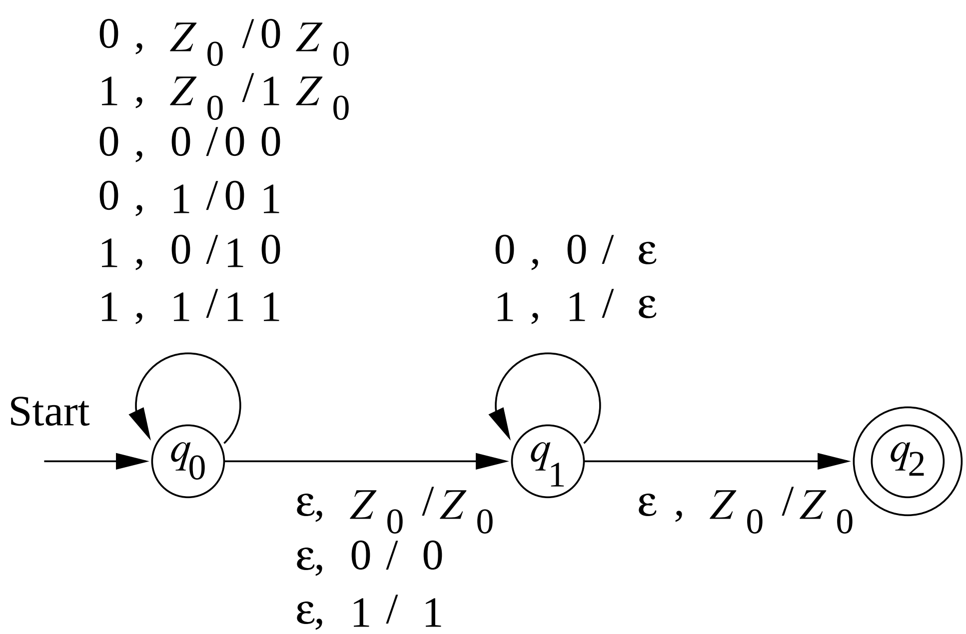
1. Guess that you are reading w. Stay in state 0, and push the input symbol onto the stack.

2. Guess that you're in the middle of ww^R. Go spontanteously to state 1.

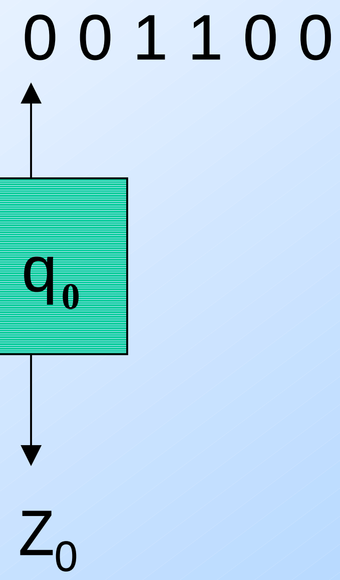
3. You're now reading the head of w^R. Compare it to the top of the stack. If they match, pop the stack, and remain in state 1. If they don't match, go to sleep.

4. If the stack is empty, go to state 2 and accept.

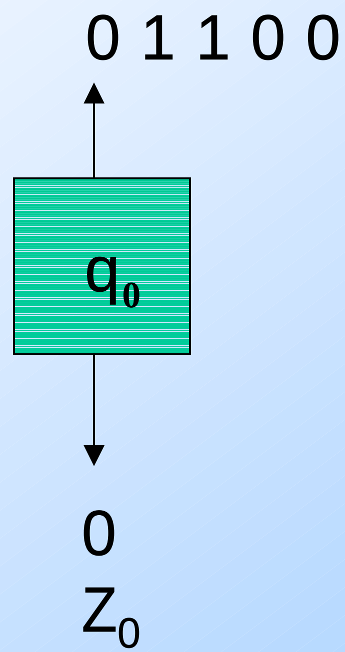
The PDA for L_{wwr} as a transition diagram:



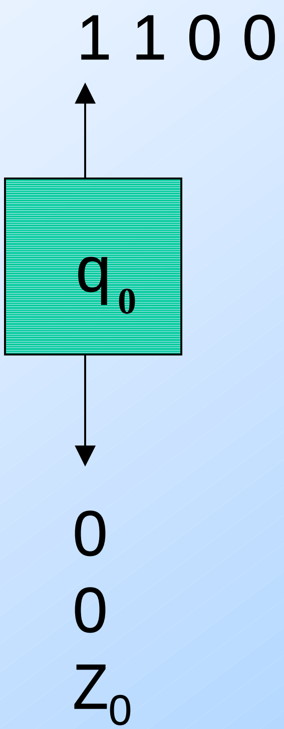
Actions of the Example PDA



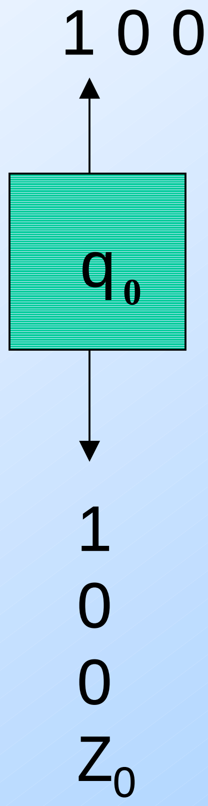
Actions of the Example PDA



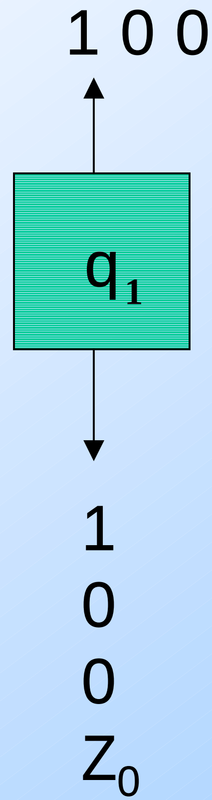
Actions of the Example PDA



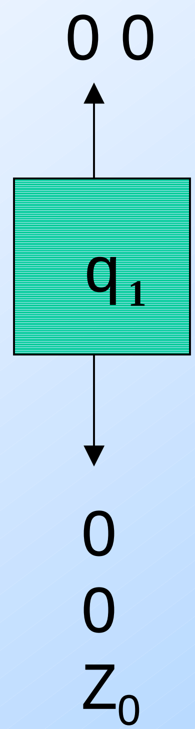
Actions of the Example PDA



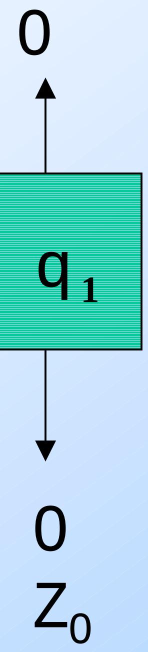
Actions of the Example PDA



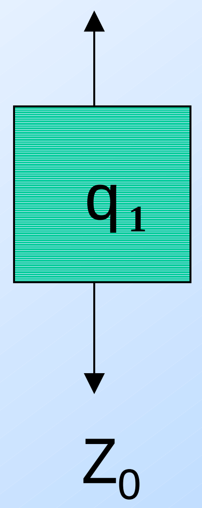
Actions of the Example PDA



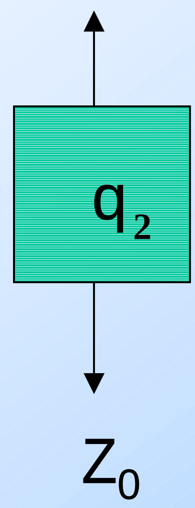
Actions of the Example PDA



Actions of the Example PDA



Actions of the Example PDA



PDA formally

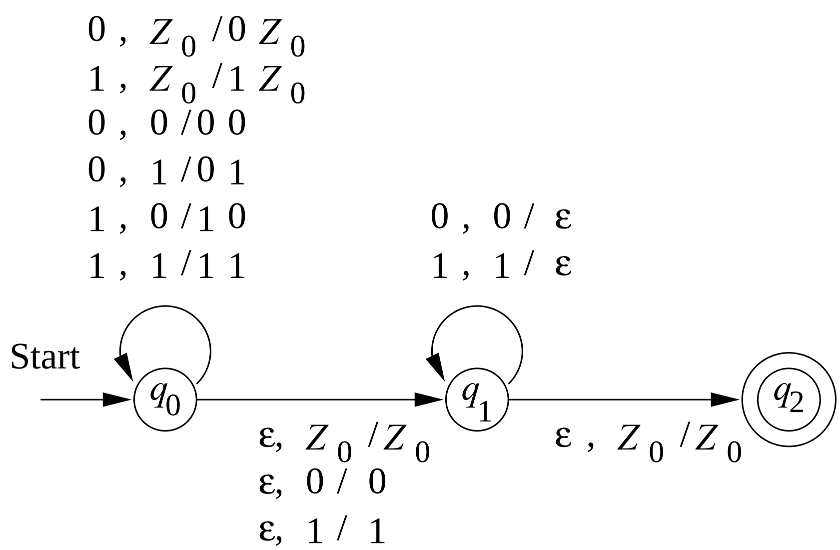
A PDA is a seven-tuple:

$$P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F),$$

where

- Q is a finite set of states,
- Σ is a finite *input alphabet*,
- Γ is a finite *stack alphabet*,
- $\delta : Q \times (\Sigma \cup \{\epsilon\}) \times \Gamma \to 2^{Q \times \Gamma^*}$ is the *transition function*,
- q_0 is the *start state*,
- $Z_0 \in \Gamma$ is the *start symbol* for the stack, and
- $F \subseteq Q$ is the set of *accepting states*.

Example: The PDA



is actually the seven-tuple

$$P = (\{q_0, q_1, q_2\}, \{0, 1\}, \{0, 1, Z_0\}, \delta, q_0, Z_0, \{q_2\}),$$

where δ is given by the following table (set brackets missing):

	$0, Z_0$	$1, Z_0$	0,0	0,1	1,0	1,1	ϵ, Z_0	$\epsilon, 0$	$\epsilon, 1$
$\rightarrow q_0$	$q_0, 0Z_0$	$q_0, 1Z_0$	$q_0, 00$	$q_0, 01$	$q_0, 10$	$q_0, 11$	q_1, Z_0	$q_1, 0$	$q_1, 1$
q_1			q_1, ϵ			q_1, ϵ	q_2, Z_0		
$\star q_2$									

Instantaneous Descriptions

A PDA goes from configuration to configuration when consuming input.

To reason about PDA computation, we use *instantaneous descriptions* of the PDA. An ID is a triple

$$(q, w, \gamma)$$

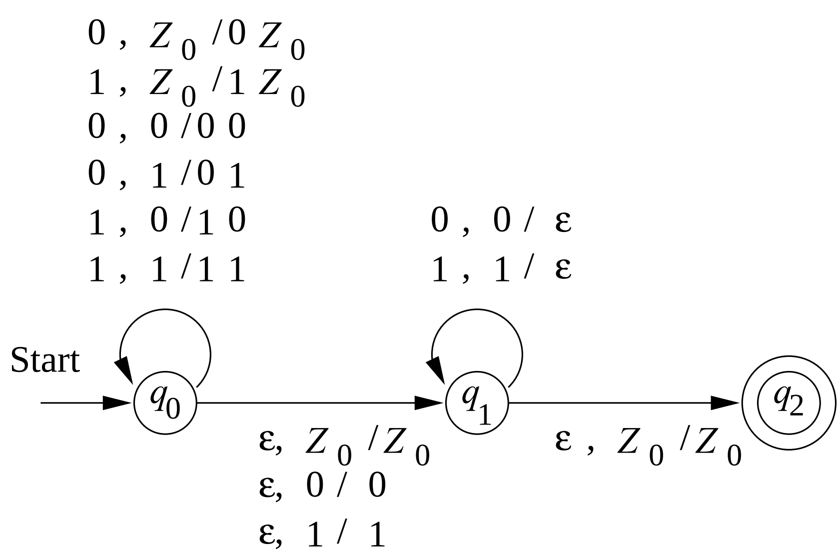
where q is the state, w the remaining input, and γ the stack contents.

Let $P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ be a PDA. Then $\forall w \in \Sigma^*, \beta \in \Gamma^*$:

$$(p, \alpha) \in \delta(q, a, X) \Rightarrow (q, aw, X\beta) \underset{\boxed{\text{yield}}}{\vdash} (p, w, \alpha\beta).$$

We define $\overset{*}{\vdash}$ to be the reflexive-transitive closure of $\vdash$.

Example: On input 1111 the PDA



has the following computation sequences:

$$
\begin{tikzcd}
(q_0, 1111, Z_0) \arrow[d] \arrow[dr] & & \\
(q_0, 111, 1Z_0) \arrow[d] \arrow[dr] & (q_1, 1111, Z_0) \arrow[r] & (q_2, 1111, Z_0) \\
(q_0, 11, 11Z_0) \arrow[d] \arrow[dr] & (q_1, 111, 1Z_0) \arrow[r] & (q_1, 11, Z_0) \arrow[d] \\
(q_0, 1, 111Z_0) \arrow[d] \arrow[dr] & (q_1, 11, 11Z_0) \arrow[dr] & (q_2, 11, Z_0) \\
(q_0, \varepsilon, 1111Z_0) \arrow[d] & (q_1, 1, 111Z_0) \arrow[d] & (q_1, 1, 1Z_0) \arrow[d] \\
(q_1, \varepsilon, 1111Z_0) & (q_1, \varepsilon, 11Z_0) & (q_1, \varepsilon, Z_0) \arrow[d] \\
 & & (q_2, \varepsilon, Z_0)
\end{tikzcd}
$$

The following properties hold:

1. If an ID sequence is a legal computation for a PDA, then so is the sequence obtained by adding an additional string at the end of component number two.

2. If an ID sequence is a legal computation for a PDA, then so is the sequence obtained by adding an additional string at the bottom of component number three.

3. If an ID sequence is a legal computation for a PDA, and some tail of the input is not consumed, then removing this tail from all ID's results in a legal computation sequence.

Theorem 6.5: $\forall w \in \Sigma^*,\ \gamma \in \Gamma^*$:

$$(q, x, \alpha) \overset{*}{\vdash} (p, y, \beta) \Rightarrow (q, xw, \alpha\gamma) \overset{*}{\vdash} (p, yw, \beta\gamma).$$

Proof: Induction on the length of the sequence to the left.

Note: If $\gamma = \epsilon$ we have proerty 1, and if $w = \epsilon$ we have property 2.

Note2: The reverse of the theorem is false.

For property 3 we have

Theorem 6.6:

$$(q, xw, \alpha) \overset{*}{\vdash} (p, yw, \beta) \Rightarrow (q, x, \alpha) \overset{*}{\vdash} (p, y, \beta).$$

Acceptance by final state

Let $P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ be a PDA. The *language accepted by P by final state* is

$$L(P) = \{w : (q_0, w, Z_0) \overset{*}{\vdash} (q, \epsilon, \alpha),\ q \in F\}.$$

Example: The PDA on slide 183 accepts exactly L_{wwr}.

Let P be the machine. We prove that $L(P) = L_{wwr}$.

($\supseteq$-*direction.*) Let $x \in L_{wwr}$. Then $x = ww^R$, and the following is a legal computation sequence

$$(q_0, ww^R, Z_0) \overset{*}{\vdash} (q_0, w^R, w^R Z_0) \vdash (q_1, w^R, w^R Z_0) \overset{*}{\vdash}$$

$$(q_1, \epsilon, Z_0) \vdash (q_2, \epsilon, Z_0).$$

($\subseteq$-*direction.*)

Observe that the only way the PDA can enter q_2 is if it is in state q_1 with **top stack symbol = z_0**

Thus it is sufficient to show that if $(q_0, x, Z_0) \overset{*}{\vdash} (q_1, \epsilon, Z_0)$ then $x = ww^R$, for some word w.

We'll show by induction on $|x|$ that

$$(q_0, x, \alpha) \overset{*}{\vdash} (q_1, \epsilon, \alpha) \;\;\Rightarrow\;\; x = ww^R.$$

Basis: If $x = \epsilon$ then x is a palindrome.

Induction: Suppose $x = a_1 a_2 \cdots a_n$, where $n > 0$, and the IH holds for shorter strings.

Ther are two moves for the PDA from ID (q_0, x, α):

Move 1: The spontaneous $(q_0, x, \alpha) \vdash (q_1, x, \alpha)$. Now $(q_1, x, \alpha) \overset{*}{\vdash} (q_1, \epsilon, \beta)$ implies that $|\beta| < |\alpha|$, which implies $\beta \neq \alpha$.

Move 2: Loop and push $(q_0, a_1 a_2 \cdots a_n, \alpha) \vdash (q_0, a_2 \cdots a_n, a_1 \alpha)$.

In this case there is a sequence

$(q_0, a_1 a_2 \cdots a_n, \alpha) \vdash (q_0, a_2 \cdots a_n, a_1 \alpha) \vdash \cdots \vdash (q_1, a_n, a_1 \alpha) \vdash (q_1, \epsilon, \alpha)$.

Thus $a_1 = a_n$ and

$$(q_0, a_2 \cdots a_n, a_1 \alpha) \overset{*}{\vdash} (q_1, a_n, a_1 \alpha).$$

By Theorem 6.6 we can remove a_n. Therefore

$$(q_0, a_2 \cdots a_{n-1}, a_1 \alpha \overset{*}{\vdash} (q_1, \epsilon, a_1 \alpha).$$

Then, by the IH $a_2 \cdots a_{n-1} = yy^R$. Then $x = a_1 yy^R a_n$ is a palindrome.

Give a final-state PDA for balanced brackets (or Dyck language): B -> BB | (B) | ε

$L_2 = \{0^m 1^n 2^p \mid m,n, p >= 0, m+n = p\}$

Acceptance by Empty Stack

Let $P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ be a PDA. The *language accepted by P by empty stack* is

$$N(P) = \{w : (q_0, w, Z_0) \overset{*}{\vdash} (q, \epsilon, \epsilon)\}.$$

Note: q can be any state.

Question: How to modify the palindrome-PDA to accept by empty stack? three ways to do it!

Give an empty-stack PDA for balanced brackets (or Dyck language): B -> BB | (B) | ε

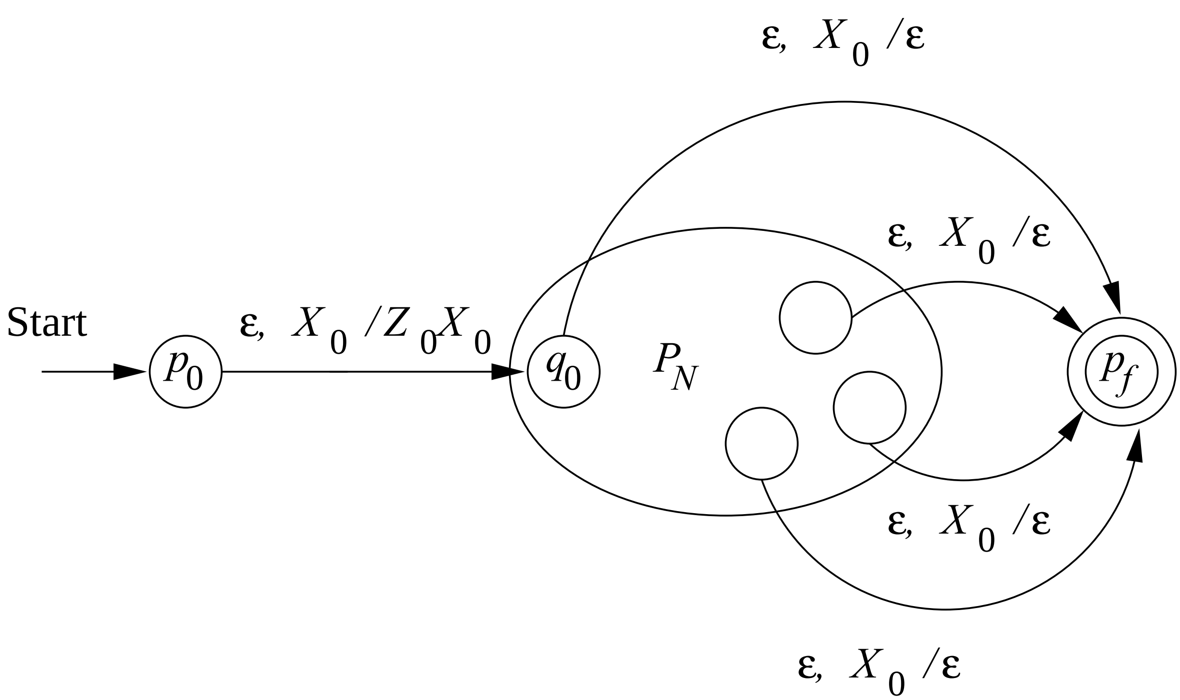
From Empty Stack to Final State

Theorem 6.9: If $L = N(P_N)$ for some PDA $P_N = (Q, \Sigma, \Gamma, \delta_N, q_0, Z_0)$, then $\exists$ PDA P_F, such that $L = L(P_F)$.

Proof: Let

$$P_F = (Q \cup \{p_0, p_f\}, \Sigma, \Gamma \cup \{X_0\}, \delta_F, p_0, X_0, \{p_f\})$$

where $\delta_F(p_0, \epsilon, X_0) = \{(q_0, Z_0X_0)\}$, and for all $q \in Q, a \in \Sigma \cup \{\epsilon\}, Y \in \Gamma : \delta_F(q, a, Y) = \delta_N(q, a, Y)$, and in addition $(p_f, \epsilon) \in \delta_F(q, \epsilon, X_0)$.



We have to show that $L(P_F) = N(P_N)$.

($\supseteq$*direction.*) Let $w \in N(P_N)$. Then

$$(q_0, w, Z_0) \overset{*}{\underset{N}{\vdash}} (q, \epsilon, \epsilon),$$

for some q. From Theorem 6.5 we get

$$(q_0, w, Z_0X_0) \overset{*}{\underset{N}{\vdash}} (q, \epsilon, X_0).$$

Since $\delta_N \subset \delta_F$ we have

$$(q_0, w, Z_0X_0) \overset{*}{\underset{F}{\vdash}} (q, \epsilon, X_0).$$

We conclude that

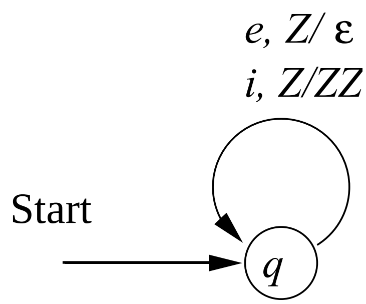
$$(p_0, w, X_0) \underset{F}{\vdash} (q_0, w, Z_0X_0) \overset{*}{\underset{F}{\vdash}} (q, \epsilon, X_0) \underset{F}{\vdash} (p_f, \epsilon, \epsilon).$$

($\subseteq$*direction.*) By inspecting the diagram.

Let's design P_N for for catching errors in strings meant to be in the *if-else*-grammar G

$$S \rightarrow \epsilon|SS|iS|iSe.$$

Here e.g. $\{ieie, iie, iei\} \subseteq$ **L(G)** and e.g. $\{ei, ieeii\} \cap$ **L(G)** $= \emptyset$. The diagram for P_N is



Note that this PDA does not really accept the complement of L(G); it gets "stuck" as soon it detects the first excess "e".

Formally,

$$P_N = (\{q\}, \{i, e\}, \{Z\}, \delta_N, q, Z),$$

where $\delta_N(q, i, Z) = \{(q, ZZ)\}$,
and $\delta_N(q, e, Z) = \{(q, \epsilon)\}$.

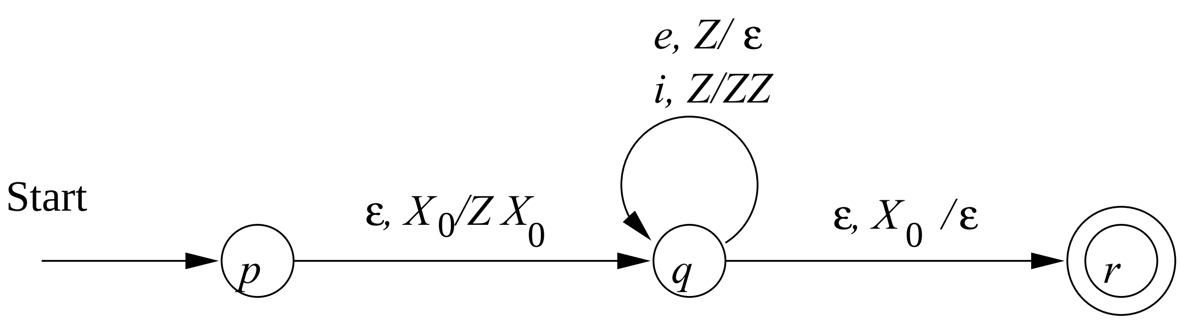
Question: Does one state suffice for empy-stack PDAs?

From P_N we can construct

$$P_F = (\{p, q, r\}, \{i, e\}, \{Z, X_0\}, \delta_F, p, X_0, \{r\}),$$

where
$\delta_F(p, \epsilon, X_0) = \{(q, ZX_0)\}$,
$\delta_F(q, i, Z) = \delta_N(q, i, Z) = \{(q, ZZ)\}$,
$\delta_F(q, e, Z) = \delta_N(q, e, Z) = \{(q, \epsilon)\}$, and
$\delta_F(q, \epsilon, X_0) = \{(r, \epsilon)\}$

The diagram for P_F is



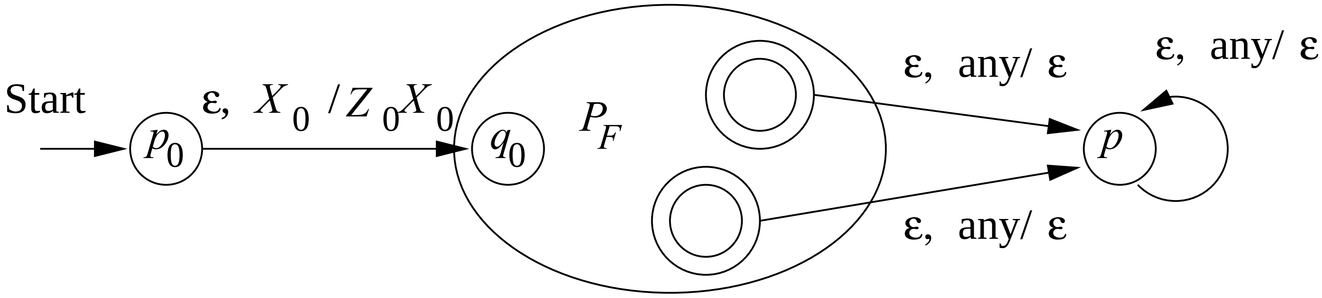
From Final State to Empty Stack

Theorem 6.11: Let $L = L(P_F)$, for some PDA $P_F = (Q, \Sigma, \Gamma, \delta_F, q_0, Z_0, F)$. Then $\exists$ PDA P_N, such that $L = N(P_N)$.

Proof: Let

$$P_N = (Q \cup \{p_0, p\}, \Sigma, \Gamma \cup \{X_0\}, \delta_N, p_0, X_0)$$

where $\delta_N(p_0, \epsilon, X_0) = \{(q_0, Z_0X_0)\}$, $\delta_N(p, \epsilon, Y) = \{(p, \epsilon)\}$, for $Y \in \Gamma \cup \{X_0\}$, and for all $q \in Q$, $a \in \Sigma \cup \{\epsilon\}, Y \in \Gamma$: $\delta_N(q, a, Y) = \delta_F(q, a, Y)$, and in addition $\forall q \in F$, and $Y \in \Gamma \cup \{X_0\}$: $(p, \epsilon) \in \delta_N(q, \epsilon, Y)$.



We have to show that $N(P_N) = L(P_F)$.

($\subseteq$-*direction*.) By inspecting the diagram.

($\supseteq$-*direction*.) Let $w \in L(P_F)$. Then

$$(q_0, w, Z_0) \overset{*}{\underset{F}{\vdash}} (q, \epsilon, \alpha),$$

for some $q \in F, \alpha \in \Gamma^*$. Since $\delta_F \subseteq \delta_N$, and Theorem 6.5 says that X_0 can be slid under the stack, we get

$$(q_0, w, Z_0X_0) \overset{*}{\underset{N}{\vdash}} (q, \epsilon, \alpha X_0).$$

The P_N can compute:

$$(p_0, w, X_0) \underset{N}{\vdash} (q_0, w, Z_0X_0) \overset{*}{\underset{N}{\vdash}} (q, \epsilon, \alpha X_0) \overset{*}{\underset{N}{\vdash}} (p, \epsilon, \epsilon).$$

Ex. Construct an empty-stack PDA for $L_3 = \{w \mid w \in \{0,1\}^*, w <> w^R\}$.

Equivalence of PDA's and CFG's

A language is

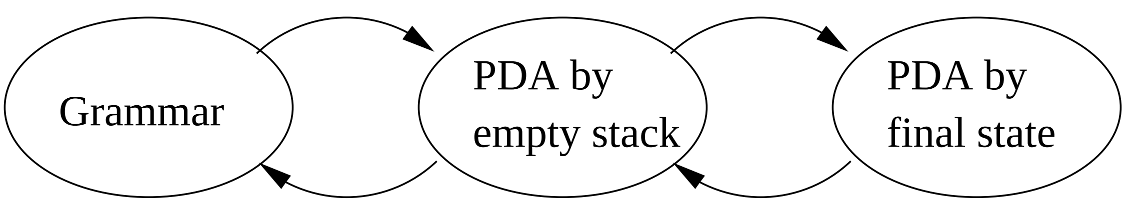
generated by a CFG

if and only if it is

accepted by a PDA by empty stack

if and only if it is

accepted by a PDA by final state



We already know how to go between null stack and final state.

From CFG's to PDA's

Given G, we construct a PDA that simulates $\overset{*}{\underset{lm}{\Rightarrow}}$.

We write left-sentential forms as

$$xA\alpha$$

where A is the leftmost variable in the form. For instance,

$$\underbrace{(a+}_{x}\underbrace{\underbrace{E}_{A}\underbrace{)}_{\alpha}}_{\text{tail}}$$

Let $xA\alpha \underset{lm}{\Rightarrow} x\beta\alpha$. This corresponds to the PDA first having consumed x and having $A\alpha$ on the stack, and then on ϵ it pops A and pushes β.

More fomally, let y, s.t. $w = xy$. Then the PDA goes non-deterministically from configuration $(q, y, A\alpha)$ to configuration $(q, y, \beta\alpha)$.

At $(q, y, \beta\alpha)$ the PDA behaves as before, unless there are terminals in the prefix of β. In that case, the PDA pops them, provided it can consume matching input.

If all guesses are right, the PDA ends up with empty stack and input.

Formally, let $G = (V, T, Q, S)$ be a CFG. Define P_G as

$$(\{q\}, T, V \cup T, \delta, q, S),$$

where

$$\delta(q, \epsilon, A) = \{(q, \beta) : A \to \beta \ \in \ Q\},$$

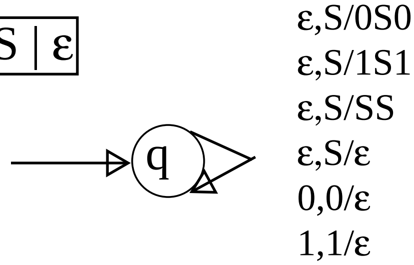
for $A \in V$, and

$$\delta(q, a, a) = \{(q, \epsilon)\},$$

for $a \in T$.

Example: On blackboard in class.

$\boxed{S \text{ -> } 0S0 \mid 1S1 \mid SS \mid \varepsilon}$



Theorem 6.13: $N(P_G) = L(G)$.

Proof:

($\supseteq$-*direction.*) Let $w \in L(G)$. Then

$$S = \gamma_1 \underset{lm}{\Rightarrow} \gamma_2 \underset{lm}{\Rightarrow} \cdots \underset{lm}{\Rightarrow} \gamma_n = w$$

Let $\gamma_i = x_i\alpha_i$. We show by induction on i that

where x_i is a string of terminals and α_i begins with a variable

$$(q, w, S) \overset{*}{\vdash} (q, y_i, \alpha_i),$$

where $w = x_i y_i$.

Basis: For $i = 1, \gamma_1 = S$. Thus $x_1 = \epsilon$, and $y_1 = w$. Clearly $(q, w, S) \overset{*}{\vdash} (q, w, S)$.

Induction: IH is $(q, w, S) \overset{*}{\vdash} (q, y_i, \alpha_i)$. We have to show that

$$(q, y_i, \alpha_i) \overset{*}{\vdash} (q, y_{i+1}, \alpha_{i+1})$$

Now α_i begins with a variable A, and we have the form

$$\underbrace{x_i A \chi}_{\gamma_i} \underset{lm}{\Rightarrow} \underbrace{x_i \;\; \beta \chi \quad}_{\gamma_{i+1}}$$

By IH $A\chi$ is on the stack, and y_i is unconsumed. From the construction of P_G it follows that we can make the move

$$(q, y_i, A\chi) \vdash (q, y_i, \beta\chi).$$

because x_{i+1} is a prefix of w

If β has a prefix of terminals, we can pop them with matching terminals in a prefix of y_i, ending up in configuration $(q, y_{i+1}, \alpha_{i+1})$, where α_{i+1} is the tail of the sentential form

$$x_{i+1}\alpha_{i+1} = \gamma_{i+1}.$$

Finally, since $\gamma_n = w$, we have $\alpha_n = \epsilon$, and $y_n = \epsilon$, and thus $(q, w, S) \overset{*}{\vdash} (q, \epsilon, \epsilon)$, i.e. $w \in N(P_G)$

($\subseteq$-*direction*.) We shall show by an induction on the length of $\overset{*}{\vdash}$, that

(♣) If $(q, x, A) \overset{*}{\vdash} (q, \epsilon, \epsilon)$, then $A \overset{*}{\Rightarrow} x$.

Basis: Length 1. Then it must be that $A \to \epsilon$ is in G, and we have $(q, \epsilon) \in \delta(q, \epsilon, A)$. Thus $A \overset{*}{\Rightarrow} \epsilon$.

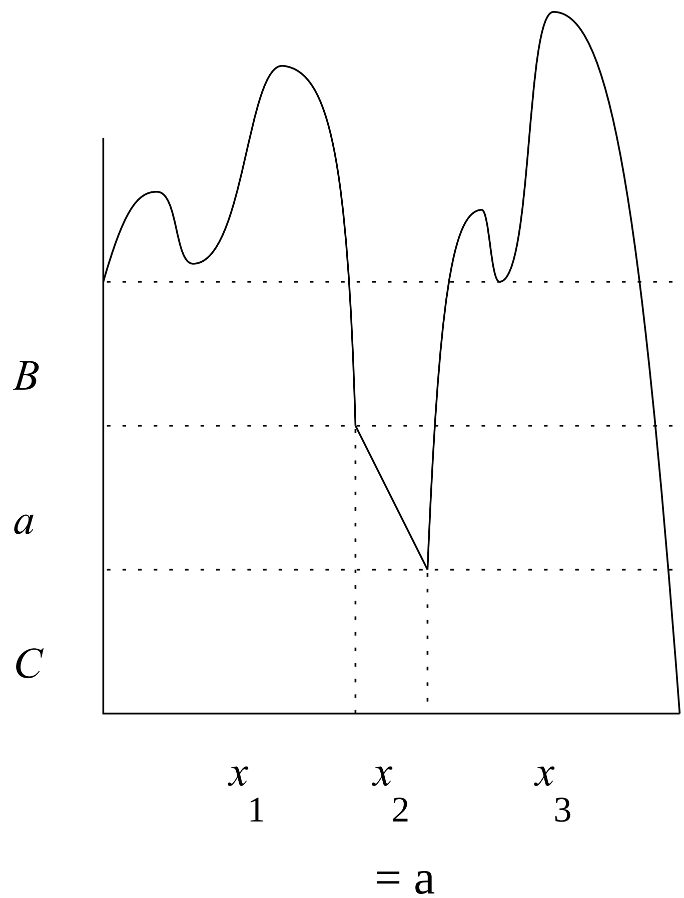
Induction: Length is $n > 1$, and the IH holds for lengths $< n$.

Since A is a variable, we must have

$$(q, x, A) \vdash (q, x, Y_1 Y_2 \cdots Y_k) \vdash \cdots \vdash (q, \epsilon, \epsilon)$$

where $A \to Y_1 Y_2 \cdots Y_k$ is in G.

We can now write x as $x_1 x_2 \cdots x_k$, according to the figure below, where $Y_1 = B, Y_2 = a$, and $Y_3 = C$.



Now we can conclude that

$$(q, x_i x_{i+1} \cdots x_k, Y_i) \overset{*}{\vdash} (q, x_{i+1} \cdots x_k, \epsilon)$$

in less than n steps, for all $i \in \{1, \ldots, k\}$. If Y_i is a variable we have by the IH and Theorem 6.6 that

$$Y_i \overset{*}{\Rightarrow} x_i$$

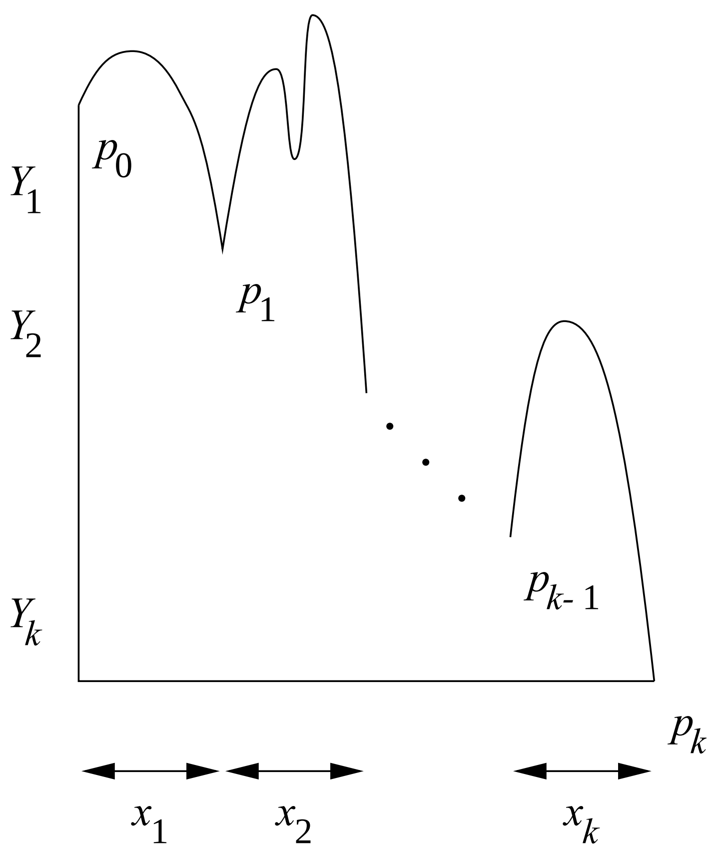
If Y_i is a terminal, we have $|x_i| = 1$, and $Y_i = x_i$. Thus $Y_i \overset{*}{\Rightarrow} x_i$ by the reflexivity of $\overset{*}{\Rightarrow}$.

Hence, $A \Longrightarrow Y_1 Y_2 \ldots Y_k \overset{*}{\Longrightarrow} x_1 x_2 \ldots x_k = x$

The claim of the theorem now follows by choosing $A = S$, and $x = w$. Suppose $w \in N(P)$. Then $(q, w, S) \overset{*}{\vdash} (q, \epsilon, \epsilon)$, and by (♣), we have $S \overset{*}{\Rightarrow} w$, meaning $w \in L(G)$.

From PDA's to CFG's

Let's look at how a PDA can consume $x = x_1 x_2 \cdots x_k$ and empty the stack.



We shall define a grammar with variables of the form $[p_{i-1} Y_i p_i]$ representing going from p_{i-1} to p_i with net effect of popping Y_i.

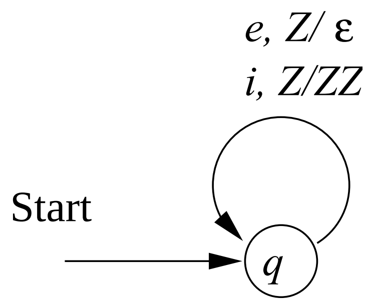
empty-stack

Formally, let $P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0)$ be a PDA. Define $G = (V, \Sigma, R, S)$, where

$$
\begin{aligned}
V &= \{[pXq] : \{p,q\} \subseteq Q, X \in \Gamma\} \cup \{S\} \\
R &= \{S \rightarrow [q_0 Z_0 p] : p \in Q\} \cup \\
&\quad \{[\mathbf{q} X r_k] \rightarrow a[\mathbf{r} Y_1 r_1] \cdots [r_{k-1} Y_k r_k] : \\
&\qquad\qquad a \in \Sigma \cup \{\epsilon\}, \\
&\qquad\qquad \{r_1, \ldots, r_k\} \subseteq Q, \\
&\qquad\qquad (\mathbf{r}, Y_1 Y_2 \cdots Y_k) \in \delta(\mathbf{q}, a, X)\}
\end{aligned}
$$

If $k = 0$, i.e. $Y_1 Y_2 \ldots Y_k = \varepsilon$, then $[\boldsymbol{q} X \boldsymbol{r}] \rightarrow$ a

Example: Let's convert



$$P_N = (\{q\}, \{i, e\}, \{Z\}, \delta_N, q, Z),$$

where $\delta_N(q, i, Z) = \{(q, ZZ)\}$,
and $\delta_N(q, e, Z) = \{(q, \epsilon)\}$ to a grammar

$$G = (V, \{i, e\}, R, S),$$

where $V = \{[qZq], S\}$, and
$R = \{[qZq] \to i[qZq][qZq], [qZq] \to e, S \to [qZq]\}$

If we replace $[qZq]$ by A we get the productions
$S \to A$ and $A \to iAA|e$.

Example: Let $P = (\{p, q\}, \{0, 1\}, \{X, Z_0\}, \delta, q, Z_0)$, where δ is given by

1. $\delta(q, 1, Z_0) = \{(q, XZ_0)\}$
2. $\delta(q, 1, X) = \{(q, XX)\}$
3. $\delta(q, 0, X) = \{(p, X)\}$
4. $\delta(q, \epsilon, X) = \{(q, \epsilon)\}$
5. $\delta(p, 1, X) = \{(p, \epsilon)\}$
6. $\delta(p, 0, Z_0) = \{(q, Z_0)\}$

What language does
this PDA accept?

to a CFG.

We get $G = (V, \{0, 1\}, R, S)$, where

$$V = \{[pXp], [pXq], [pZ_0p], [pZ_0q], S\}$$

$[qXq], [qXp], [qZ_0p], [qZ_0q]$

and the productions in R are

$$S \rightarrow [qZ_0q] | [qZ_0p]$$

From transition (1):

$$[qZ_0q] \rightarrow 1[qXq][qZ_0q]$$
$$[qZ_0q] \rightarrow 1[qXp][pZ_0q]$$
$$[qZ_0p] \rightarrow 1[qXq][qZ_0p]$$
$$[qZ_0p] \rightarrow 1[qXp][pZ_0p]$$

From transition (2):

$$[qXq] \rightarrow 1[qXq][qXq]$$
$$[qXq] \rightarrow 1[qXp][pXq]$$
$$[qXp] \rightarrow 1[qXq][qXp]$$
$$[qXp] \rightarrow 1[qXp][pXp]$$

From transition (3):

$[qXq] \rightarrow 0[pXq]$
$[qXp] \rightarrow 0[pXp]$

From transition (4):

$[qXq] \rightarrow \epsilon$

From transition (5):

$[pXp] \rightarrow 1$

From transition (6):

$[pZ_0q] \rightarrow 0[qZ_0q]$
$[pZ_0p] \rightarrow 0[qZ_0p]$

Theorem 6.14: Let G be constructed from a PDA P as above. Then $L(G) = N(P)$

Proof:

($\supseteq$-*direction.*) We shall show by an induction on the length of the sequence $\overset{*}{\vdash}$ that

(♠) If $(q, w, X) \overset{*}{\vdash} (p, \epsilon, \epsilon)$ then $[qXp] \overset{*}{\Rightarrow} w$.

Basis: Length 1. Then w is an a or ϵ, and $(p, \epsilon) \in \delta(q, w, X)$. By the construction of G we have $[qXp] \rightarrow w$ and thus $[qXp] \overset{*}{\Rightarrow} w$.

Induction: Length is $n > 1$, and ♠ holds for lengths $< n$. We must have

$$(q, w, X) \vdash (r_0, x, Y_1 Y_2 \cdots Y_k) \vdash \cdots \vdash (p, \epsilon, \epsilon),$$

where $w = ax$ or $w = \epsilon x$. It follows that $(r_0, Y_1 Y_2 \cdots Y_k) \in \delta(q, a, X)$. Then we have a production

$$[qXr_k] \to a[r_0 Y_1 r_1] \cdots [r_{k-1} Y_k r_k],$$

for all $\{r_1, \ldots, r_k\} \subset Q$.

We may now choose r_i to be the state in the sequence $\overset{*}{\vdash}$ when Y_i is popped. Let $x = w_1 w_2 \cdots w_k$, where w_i is consumed while Y_i is popped. Then

$$(r_{i-1}, w_i, Y_i) \overset{*}{\vdash} (r_i, \epsilon, \epsilon).$$

Note that $r_k = p$

By the IH we get

$$[r_{i-1} Y r_i] \overset{*}{\Rightarrow} w_i$$

We then get the following derivation sequence:

$\boxed{r_k = p}$

$$[qXr_k] \Rightarrow a[r_0Y_1r_1]\cdots[r_{k-1}Y_kr_k] \stackrel{*}{\Rightarrow}$$

$$aw_1[r_1Y_2r_2][r_2Y_3r_3]\cdots[r_{k-1}Y_kr_k] \stackrel{*}{\Rightarrow}$$

$$aw_1w_2[r_2Y_3r_3]\cdots[r_{k-1}Y_kr_k] \stackrel{*}{\Rightarrow}$$

$$\cdots$$

$$aw_1w_2\cdots w_k = w \quad = ax$$

($\supseteq$-*direction.*) We shall show by an induction on the length of the derivation $\stackrel{*}{\Rightarrow}$ that

($\heartsuit$) If $[qXp] \stackrel{*}{\Rightarrow} w$ then $(q, w, X) \stackrel{*}{\vdash} (p, \epsilon, \epsilon)$

Basis: One step. Then we have a production $[qXp] \rightarrow w$. From the construction of G it follows that $(p, \epsilon) \in \delta(q, a, X)$, where $w = a$. But then $(q, w, X) \stackrel{*}{\vdash} (p, \epsilon, \epsilon)$.

Induction: Length of $\stackrel{*}{\Rightarrow}$ is $n > 1$, and $\heartsuit$ holds for lengths $< n$. Then we must have

$\boxed{r_k = p}$

$$[qXr_k] \Rightarrow a[r_0Y_1r_1][r_1Y_2r_2] \cdots [r_{k-1}Y_kr_k] \stackrel{*}{\Rightarrow} w$$

We can break w into $aw_1 \cdots w_k$ such that $[r_{i-1}Y_ir_i] \stackrel{*}{\Rightarrow} w_i$. From the IH we get

$$(r_{i-1}, w_i, Y_i) \stackrel{*}{\vdash} (r_i, \epsilon, \epsilon)$$

From Theorem 6.5 we get

$$(r_{i-1}, w_i w_{i+1} \cdots w_k, Y_i Y_{i+1} \cdots Y_k) \overset{*}{\vdash} (r_i, w_{i+1} \cdots w_k, Y_{i+1} \cdots Y_k)$$

Since this holds for all $i \in \{1, \ldots, k\}$, we get

$$(q, aw_1 w_2 \cdots w_k, X) \vdash$$

$$(r_0, w_1 w_2 \cdots w_k, Y_1 Y_2 \cdots Y_k) \overset{*}{\vdash}$$

$$(r_1, w_2 \cdots w_k, Y_2 \cdots Y_k) \overset{*}{\vdash}$$

$$(r_2, w_3 \cdots w_k, Y_3 \cdots Y_k) \overset{*}{\vdash}$$

$$(p, \epsilon, \epsilon).$$

since $(r_0, Y_1 Y_2 \ldots Y_k) \in \delta(q,a,X)$

$p = r_k$

Q1. Can you give a 1-state empty stack PDA for $L_1 = \{ 0^n 1^n \mid n >= 0\}$?

Q2: How to decide if a PDA M accepts a string w?

Deterministic PDA's

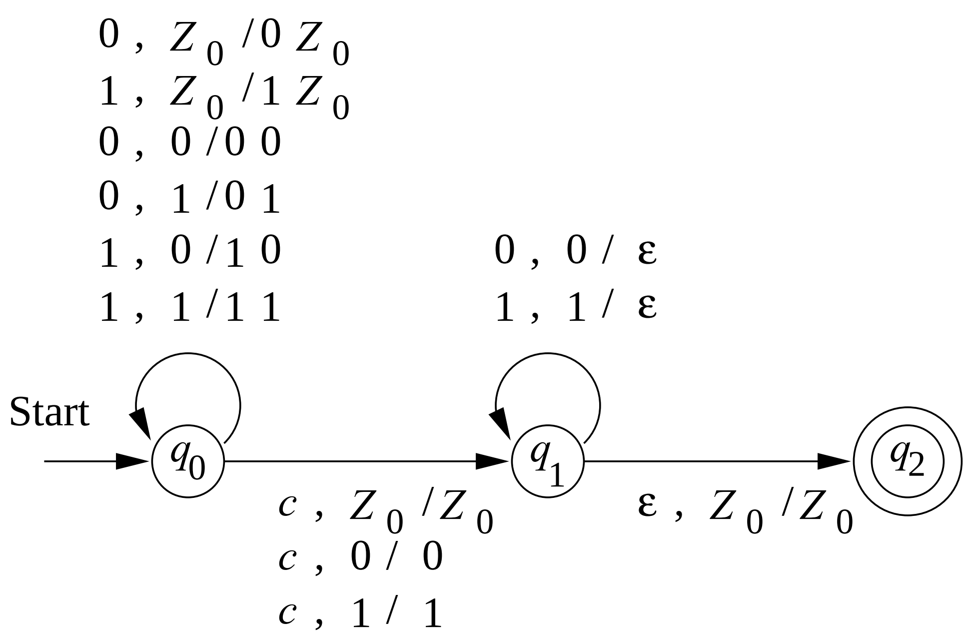
A PDA $P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ is *deterministic* iff

1. $\delta(q, \overset{\varepsilon}{a}, X)$ is always empty or a singleton.
2. If $\delta(q, a, X)$ is nonempty, then $\delta(q, \epsilon, X)$ must be empty.

Example: Let us define

$$L_{wcwr} = \{wcw^R : w \in \{0,1\}^*\}$$

Then L_{wcwr} is recognized by the following DPDA



We'll show that Regular$\subset L$(DPDA) $\subset$ CFL

Theorem 6.17: If L is regular, then $L = L(P)$ for some DPDA P.

Proof: Since L is regular there is a DFA A s.t. $L = L(A)$. Let

$$A = (Q, \Sigma, \delta_A, q_0, F)$$

We define the DPDA

$$P = (Q, \Sigma, \{Z_0\}, \delta_P, q_0, Z_0, F),$$

where

$$\delta_P(q, a, Z_0) = \{(\delta_A(q, a), Z_0)\},$$

for all $p, q \in Q$, and $a \in \Sigma$.

An easy induction (do it!) on $|w|$ gives

$$(q_0, w, Z_0) \overset{*}{\vdash} (p, \epsilon, Z_0) \Leftrightarrow \hat{\delta_A}(q_0, w) = p$$

The theorem then follows (why?)

What about DPDA's that accept by null stack?

They can recognize only CFL's with the prefix property.

A language L has the *prefix property* if there are no two distinct strings in L, such that one is a prefix of the other.

Example: L_{wcwr} has the prefix property.

Example: $\{0\}^*$ does not have the prefix property.

Theorem 6.19: L is $N(P)$ for some DPDA P if and only if L has the prefix property and L is $L(P')$ for some DPDA P'.

Proof: Homework

- We have seen that Regular$\subseteq L$(DPDA).

- $L_{wcwr} \in L$(DPDA)$\setminus$ Regular

- Are there languages in CFL$\setminus L$(DPDA).

  Yes, for example L_{wwr}.

- What about DPDA's and Ambiguous Grammars?

  L_{wwr} has unamb. grammar $S \rightarrow 0S0|1S1|\epsilon$
  but is not L(DPDA).

But LL(k) languages are in L(DPDA)!

For the converse we have

Theorem 6.20: If $L = N(P)$ for some DPDA P, then L has an unambiguous CFG.

Proof: By inspecting the proof of Theorem 6.14 we see that if the construction is applied to a DPDA the result is a CFG with unique leftmost derivations.

Theorem 6.20 can actually be strengthened as follows

Theorem 6.21: If $L = L(P)$ for some DPDA P, then L has an unambiguous CFG.

Proof: Let \$ be a symbol outside the alphabet of L, and let $L' = L\$$.

It is easy to see that L' has the prefix property.

By Theorem 6.20 we have $L' = N(P')$ for some DPDA P'.

By Theorem 6.20 $N(P')$ can be generated by an unambiguous CFG G'

Modify G' into G, s.t. $L(G) = L$, by adding the production

$$\$ \rightarrow \epsilon$$

Since G' has unique leftmost derivations, G also has unique lm's, since the only new thing we're doing is adding derivations

$$w\$ \underset{lm}{\Rightarrow} w$$

to the end.

L(DPDA) are called deterministic CFLs, equivalent to LR(k) languages.

Properties of CFL's

• *Simplification* of CFG's. This makes life easier, since we can claim that if a language is CF, then it has a grammar of a special form.

• *Pumping Lemma for CFL's.* Similar to the regular case.

• *Closure properties.* Some, but not all, of the closure properties of regular languages carry over to CFL's.

• *Decision properties.* We can test for membership and emptiness, but for instance, equivalence of CFL's is undecidable.

Chomsky Normal Form

We want to show that every CFL (without ϵ) is generated by a CFG where all productions are of the form

$$A \to BC, \text{ or } A \to a$$

where A, B, and C are variables, and a is a terminal. This is called CNF, and to get there we have to

1. Eliminate *useless symbols*, those that do not appear in any derivation $S \stackrel{*}{\Rightarrow} w$, for start symbol S and terminal w.

2. Eliminate *ϵ-productions*, that is, productions of the form $A \to \epsilon$.

3. Eliminate *unit productions*, that is, productions of the form $A \to B$, where A and B are variables.

Eliminating Useless Symbols

- A symbol X is *useful* for a grammar $G = (V, T, P, S)$, if there is a derivation

$$S \overset{*}{\underset{G}{\Rightarrow}} \alpha X \beta \overset{*}{\underset{G}{\Rightarrow}} w$$

for a teminal string w. Symbols that are not useful are called *useless*.

- A symbol X is *generating* if $X \overset{*}{\underset{G}{\Rightarrow}} w$, for some $w \in T^*$

- A symbol X is *reachable* if $S \overset{*}{\underset{G}{\Rightarrow}} \alpha X \beta$, for some $\{\alpha, \beta\} \subseteq (V \cup T)^*$

It turns out that if we eliminate non-generating symbols first, and then non-reachable ones, we will be left with only useful symbols.

Example: Let G be

$$S \to AB|a, \; A \to b$$

S and A are generating, B is not. If we eliminate B we have to eliminate $S \to AB$, leaving the grammar

$$S \to a, A \to b$$

Now only S and a are reachable. Eliminating A and b leaves us with

$$S \to a$$

with language $\{a\}$.

OTH, if we eliminate non-reachable symbols first, we find that all symbols are reachable. From

$$S \to AB|a, \; A \to b$$

we then eliminate B as non-generating, and are left with

$$S \to a, \; A \to b$$

that still contains useless symbols

Theorem 7.2: Let $G = (V, T, P, S)$ be a CFG such that $L(G) \neq \emptyset$. Let $G_1 = (V_1, T_1, P_1, S)$ be the grammar obtained by

1. Eliminating all nongenerating symbols and the productions they occur in. Let the new grammar be $G_2 = (V_2, T_2, P_2, S)$.

2. Eliminate from G_2 all nonreachable symbols and the productions they occur in.

Then G_1 has no useless symbols, and $L(G_1) = L(G)$.

Proof: We first prove that G_1 has no useless symbols:

Let X remain in $V_1 \cup T_1$. Thus $X \stackrel{*}{\Rightarrow} w$ in G , for some $w \in T^*$. Moreover, every symbol used in this derivation is also generating. Thus $X \stackrel{*}{\Rightarrow} w$ in G_2 also. But this is not enough!

Since X was not eliminated in step 2, there are α and β, such that $S \stackrel{*}{\Rightarrow} \alpha X \beta$ in G_2. Furthermore, every symbol used in this derivation is also reachable, so $S \stackrel{*}{\Rightarrow} \alpha X \beta$ in G_1.

Now every symbol in $\alpha X \beta$ is reachable and in $V_2 \cup T_2 \supseteq V_1 \cup T_1$, so each of them is generating in G_2.

The terminal derivation $\alpha X \beta \stackrel{*}{\Rightarrow} xwy$ in G_2 involves only symbols that are reachable from S, because they are reached from symbols in $\alpha X \beta$. Thus the terminal derivation is also a derviation in G_1, i.e.,

$$S \stackrel{*}{\Rightarrow} \alpha X \beta \stackrel{*}{\Rightarrow} xwy$$

in G_1.

We then show that $L(G_1) = L(G)$.

Since $P_1 \subseteq P$, we have $L(G_1) \subseteq L(G)$.

Then, let $w \in L(G)$. Thus $S \overset{*}{\underset{G}{\Rightarrow}} w$. Each symbol is this derivation is evidently both reachable and generating, so this is also a derivation of G_1.

Thus $w \in L(G_1)$.

We have to give algorithms to compute the generating and reachable symbols of $G = (V, T, P, S)$.

The generating symbols $g(G)$ are computed by the following closure algorithm:

Basis: $g(G) == T$

Induction: If $\alpha \in g(G)^*$ and $X \to \alpha \in P$, then $g(G) == g(G) \cup \{X\}$.

Example: Let G be $S \to AB|a, \ A \to b$

Then first $g(G) == \{a, b\}$.

Since $S \to a$ we put S in $g(G)$, and because $A \to b$ we add A also, and that's it.

Theorem 7.4: At saturation, $g(G)$ contains all and only the generating symbols of G.

Proof:

We'll show in class by an induction on the stage in which a symbol X is added to $g(G)$ that X is indeed generating.

Then, suppose that X is generating. Thus $X \underset{G}{\overset{*}{\Rightarrow}} w$, for some $w \in T^*$. We prove by induction on this derivation that $X \in g(G)$.

Basis: Zero Steps. Then X is added in the basis of the closure algo.

Induction: The derivation takes $n > 0$ steps. Let the first production used be $X \to \alpha$. Then

$$X \Rightarrow \alpha \overset{*}{\Rightarrow} w$$

and $\alpha \overset{*}{\Rightarrow} w$ in less than n steps and by the IH $\alpha \in g(G)^*$. From the inductive part of the algo it follows that $X \in g(G)$.

The set of reachable symbols $r(G)$ of $G = (V, T, P, S)$ is computed by the following closure algorithm:

Basis: $r(G) == \{S\}$.

Induction: If variable $A \in r(G)$ and $A \rightarrow \alpha \in P$ then add all symbols in α to $r(G)$

Example: Let G be $S \rightarrow AB|a, \; A \rightarrow b$

Then first $r(G) == \{S\}$.

Based on the first production we add $\{A, B, a\}$ to $r(G)$.

Based on the second production we add $\{b\}$ to $r(G)$ and that's it.

Theorem 7.6: At saturation, $r(G)$ contains all and only the reachable symbols of G.

Proof: Homework.

Eliminating ϵ-Productions

We shall prove that if L is CF, then $L \setminus \{\epsilon\}$ has a grammar without ϵ-productions.

Variable A is said to be *nullable* if $A \stackrel{*}{\Rightarrow} \epsilon$.

Let A be nullable. We'll then replace a rule like

$$A \rightarrow BAD$$

with

$$A \rightarrow BAD, \ A \rightarrow BD$$

and delete any rules with body ϵ.

We'll compute $n(G)$, the set of nullable symbols of a grammar $G = (V, T, P, S)$ as follows:

Basis: $n(G) == \{A : A \rightarrow \epsilon \in P\}$

Induction: If $\{C_1, C_2, \cdots, C_k\} \subseteq n(G)$ and $A \rightarrow C_1 C_2 \cdots C_k \in P$, then $n(G) == n(G) \cup \{A\}$.

Theorem 7.7: At saturation, $n(G)$ contains all and only the nullable symbols of G.

Proof: Easy induction in both directions.

Once we know the nullable symbols, we can transform G into G_1 as follows:

- For each $A \rightarrow X_1 X_2 \cdots X_k \in P$ with $m \leq k$ nullable symbols, replace it by 2^m rules, one with each sublist of the nullable symbols absent.

Exeption: If $m = k$ we don't delete all m nullable symbols.

- Delete all rules of the form $A \rightarrow \epsilon$.

Example: Let G be

$$S \to AB,\ A \to aAA|\epsilon,\ B \to bBB|\epsilon$$

Now $n(G) = \{A, B, S\}$. The first rule will become

$$S \to AB|A|B$$

the second

$$A \to aAA|aA|aA|a$$

the third

$$B \to bBB|bB|bB|b$$

We then delete rules with ϵ-bodies, and end up with grammar G_1 :

$$S \to AB|A|B,\ A \to aAA|aA|a,\ B \to bBB|bB|b$$

Theorem 7.9: $L(G_1) = L(G) \setminus \{\epsilon\}$.

Proof: We'll prove the stronger statement:

($\sharp$) $A \stackrel{*}{\Rightarrow} w$ in G_1 if and only if $w \neq \epsilon$ and $A \stackrel{*}{\Rightarrow} w$ in G.

$\subseteq$-*direction:* Suppose $A \stackrel{*}{\Rightarrow} w$ in G_1. Then clearly $w \neq \epsilon$ (Why?). We'll show by an induction on the length of the derivation that $A \stackrel{*}{\Rightarrow} w$ in G also.

Basis: One step. Then there exists $A \to w$ in G_1. From the construction of G_1 it follows that there exists $A \to \alpha$ in G, where α is w plus some nullable variables interspersed. Then

$$A \Rightarrow \alpha \stackrel{*}{\Rightarrow} w$$

in G.

Induction: Derivation takes $n > 1$ steps. Then

$$A \Rightarrow X_1 X_2 \cdots X_k \stackrel{*}{\Rightarrow} w \text{ in } G_1$$

and the first derivation is based on a production

$$A \rightarrow Y_1 Y_2 \cdots Y_m \qquad \text{in } G$$

where $m \geq k$, some Y_i's are X_j's and the other are nullable symbols of G.

Furthermore, $w = w_1 w_2 \cdots w_k$, and $X_i \stackrel{*}{\Rightarrow} w_i$ in G_1 in less than n steps. By the IH we have $X_i \stackrel{*}{\Rightarrow} w_i$ in G. Now we get

$$A \underset{G}{\Rightarrow} Y_1 Y_2 \cdots Y_m \underset{G}{\stackrel{*}{\Rightarrow}} X_1 X_2 \cdots X_k \underset{G}{\stackrel{*}{\Rightarrow}} w_1 w_2 \cdots w_k = w$$

$\supseteq$*-direction:* Let $A \overset{*}{\underset{G}{\Rightarrow}} w$, and $w \neq \epsilon$. We'll show by induction of the length of the derivation that $A \overset{*}{\Rightarrow} w$ in G_1.

Basis: Length is one. Then $A \rightarrow w$ is in G, and since $w \neq \epsilon$ the rule is in G_1 also.

Induction: Derivation takes $n > 1$ steps. Then it looks like

$$A \underset{G}{\Rightarrow} Y_1 Y_2 \cdots Y_m \overset{*}{\underset{G}{\Rightarrow}} w$$

Now $w = w_1 w_2 \cdots w_m$, and $Y_i \overset{*}{\underset{G}{\Rightarrow}} w_i$ in less than n steps.

Let $X_1 X_2 \cdots X_k$ be those Y_j's in order, such that $w_j \neq \epsilon$. Then $A \rightarrow X_1 X_2 \cdots X_k$ is a rule in G_1.

Now $X_1 X_2 \cdots X_k \overset{*}{\underset{G}{\Rightarrow}} w$ (Why?)

Each $X_j/Y_j \overset{*}{\underset{G}{\Rightarrow}} w_j$ in less than n steps, so by IH we have that if $w_j \neq \epsilon$ then $Y_j \overset{*}{\Rightarrow} w_j$ in G_1. Thus

$$A \Rightarrow X_1 X_2 \cdots X_k \overset{*}{\Rightarrow} w \text{ in } G_1$$

The claim of the theorem now follows from statement (♯) on slide 238 by choosing $A = S$.

Eliminating Unit Productions

$$A \rightarrow B$$

is a *unit* production, whenever A and B are variables.

Unit productions can be eliminated.

Let's look at grammar

$I \rightarrow a \mid b \mid Ia \mid Ib \mid I0 \mid I1$
$F \rightarrow I \mid (E)$
$T \rightarrow F \mid T * F$
$E \rightarrow T \mid E + T$

It has unit productions $E \rightarrow T$, $T \rightarrow F$, and $F \rightarrow I$

We'll expand rule $E \rightarrow T$ and get rules

$$E \rightarrow F,\ E \rightarrow T * F$$

We then expand $E \rightarrow F$ and get

$$E \rightarrow I|(E)|T * F$$

Finally we expand $E \rightarrow I$ and get

$$E \rightarrow a \mid b \mid Ia \mid Ib \mid I0 \mid I1 \mid (E) \mid T * F$$

The expansion method works as long as there are no cycles in the rules, as e.g. in

$$A \rightarrow B,\ B \rightarrow C,\ C \rightarrow A$$

The following method based on *unit pairs* will work for all grammars.

(A, B) is a *unit pair* if $A \stackrel{*}{\Rightarrow} B$ using unit productions only.

Note: In $A \to BC,\ C \to \epsilon$ we have $A \stackrel{*}{\Rightarrow} B$, but not using unit productions only.

To compute $u(G)$, the set of all unit pairs of $G = (V, T, P, S)$ we use the following closure algorithm

Basis: $u(G) == \{(A, A) : A \in V\}$

Induction: If $(A, B) \in u(G)$ and $B \to C \in P$ then add (A, C) to $u(G)$.

Theorem: At saturation, $u(G)$ contains all and only the unit pair of G.

Proof: Easy.

Given $G = (V, T, P, S)$ we can construct $G_1 = (V, T, P_1, S)$ that doesn't have unit productions, and such that $L(G_1) = L(G)$ by setting

$$P_1 = \{A \to \alpha : \alpha \notin V, B \to \alpha \in P, (A, B) \in u(G)\}$$

Example: For the grammar of slide 242 we get

Pair	Productions
(E, E)	$E \to E + T$
(E, T)	$E \to T * F$
(E, F)	$E \to (E)$
(E, I)	$E \to a \mid b \mid Ia \mid Ib \mid I0 \mid I1$
(T, T)	$T \to T * F$
(T, F)	$T \to (E)$
(T, I)	$T \to a \mid b \mid Ia \mid Ib \mid I0 \mid I1$
(F, F)	$F \to (E)$
(F, I)	$F \to a \mid b \mid Ia \mid Ib \mid I0 \mid I1$
(I, I)	$I \to a \mid b \mid Ia \mid Ib \mid I0 \mid I1$

The resulting grammar is equivalent to the original one (proof omitted).

Summary

To "clean up" a grammar we can

1. Eliminate ϵ-productions

2. Eliminate unit productions

3. Eliminate useless symbols

in this order.

This cannot be done earlier due to the removal of ε–productions and unit productions.

Chomsky Normal Form, CNF

We shall show that every nonempty CFL without ϵ has a grammar G without useless symbols, and such that every production is of the form

- $A \rightarrow BC$, where $\{A, B, C\} \subseteq V$, or
- $A \rightarrow a$, where $A \in V$, and $a \in T$.

To achieve this, start with any grammar for the CFL, and

1. "Clean up" the grammar.
2. Arrange that all bodies of length 2 or more consists of only variables.
3. Break bodies of length 3 or more into a cascade of two-variable-bodied productions.

• For step 2, for every terminal a that appears in a body of length ≥ 2, create a new variable, say A, and replace a by A in all bodies.

Then add a new rule $A \rightarrow a$.

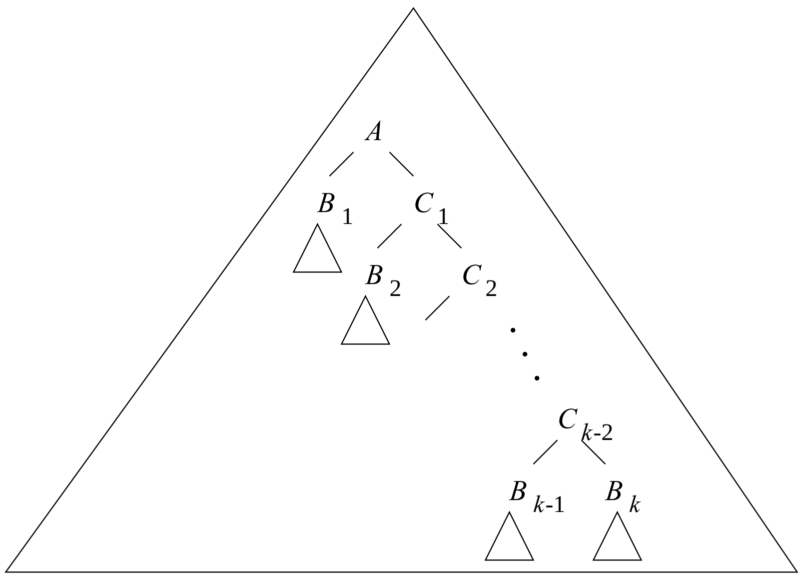
• For step 3, for each rule of the form

$$A \rightarrow B_1 B_2 \cdots B_k,$$

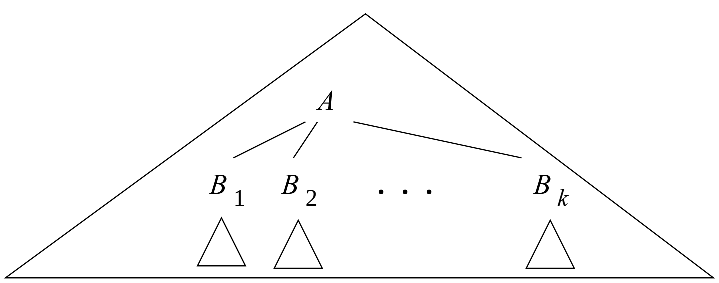
$k \geq 3$, introduce new variables $C_1, C_2, \ldots C_{k-2}$, and replace the rule with

$$\begin{array}{rcl}
A & \rightarrow & B_1 C_1 \\
C_1 & \rightarrow & B_2 C_2 \\
 & \cdots & \\
C_{k-3} & \rightarrow & B_{k-2} C_{k-2} \\
C_{k-2} & \rightarrow & B_{k-1} B_k
\end{array}$$

Illustration of the effect of step 3



(a)



(b)

Example of CNF conversion

Let's start with the grammar (step 1 already done)

$$E \rightarrow E + T \mid T * F \mid (E) \mid a \mid b \mid Ia \mid Ib \mid I0 \mid I1$$
$$T \rightarrow T * F \mid (E) \mid a \mid b \mid Ia \mid Ib \mid I0 \mid I1$$
$$F \rightarrow (E) \mid a \mid b \mid Ia \mid Ib \mid I0 \mid I1$$
$$I \rightarrow a \mid b \mid Ia \mid Ib \mid I0 \mid I1$$

For step 2, we need the rules
$A \rightarrow a, B \rightarrow b, Z \rightarrow 0, O \rightarrow 1$
$P \rightarrow +, M \rightarrow *, L \rightarrow (, R \rightarrow)$
and by replacing we get the grammar

$$E \rightarrow EPT \mid TMF \mid LER \mid a \mid b \mid IA \mid IB \mid IZ \mid IO$$
$$T \rightarrow TMF \mid LER \mid a \mid b \mid IA \mid IB \mid IZ \mid IO$$
$$F \rightarrow LER \mid a \mid b \mid IA \mid IB \mid IZ \mid IO$$
$$I \rightarrow a \mid b \mid IA \mid IB \mid IZ \mid IO$$
$$A \rightarrow a, B \rightarrow b, Z \rightarrow 0, O \rightarrow 1$$
$$P \rightarrow +, M \rightarrow *, L \rightarrow (, R \rightarrow)$$

For step 3, we replace

$E \rightarrow EPT$ by $E \rightarrow EC_1, C_1 \rightarrow PT$

$E \rightarrow TMF, T \rightarrow TMF$ by
$E \rightarrow TC_2, T \rightarrow TC_2, C_2 \rightarrow MF$

$E \rightarrow LER, T \rightarrow LER, F \rightarrow LER$ by
$E \rightarrow LC_3, T \rightarrow LC_3, F \rightarrow LC_3, C_3 \rightarrow ER$

The final CNF grammar is

$E \rightarrow EC_1 \mid TC_2 \mid LC_3 \mid a \mid b \mid IA \mid IB \mid IZ \mid IO$
$T \rightarrow TC_2 \mid LC_3 \mid a \mid b \mid IA \mid IB \mid IZ \mid IO$
$F \rightarrow LC_3 \mid a \mid b \mid IA \mid IB \mid IZ \mid IO$
$I \rightarrow a \mid b \mid IA \mid IB \mid IZ \mid IO$
$C_1 \rightarrow PT, C_2 \rightarrow MF, C_3 \rightarrow ER$
$A \rightarrow a, B \rightarrow b, Z \rightarrow 0, O \rightarrow 1$
$P \rightarrow +, M \rightarrow *, L \rightarrow (, R \rightarrow)$

The Pumping Lemma for CFL's

Statement

Applications

Intuition

- Recall the pumping lemma for regular languages.
- It told us that if there was a string long enough to cause a cycle in the DFA for the language, then we could “pump” the cycle and discover an infinite sequence of strings that had to be in the language.

Intuition – (2)

- For CFL's the situation is a little more complicated.
- We can always find two pieces of any sufficiently long string to "pump" in tandem.
  - That is: if we repeat each of the two pieces the same number of times, we get another string in the language.

Statement of the CFL Pumping Lemma

For every context-free language L

There is an integer n, such that

For every string z in L of length $\geq$ n

There exists z = uvwxy such that:

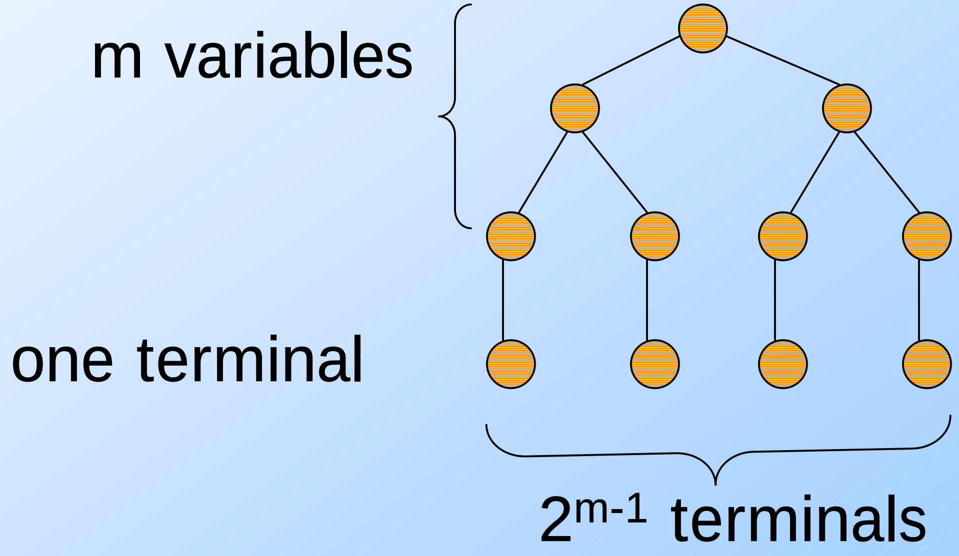
1. |vwx| $\leq$ n.
2. |vx| > 0.
3. For all i $\geq$ 0, uv^iwx^iy is in L.

Proof of the Pumping Lemma

- Start with a CNF grammar for $L - \{\epsilon\}$.
- Let the grammar have m variables.
- Pick $n = 2^m$.
- Let $z \in L$ and $|z| \geq n$.
- We claim ("*Lemma 1*") that a parse tree with yield z must have a path of length m+2 nodes or more.

Proof of Lemma 1

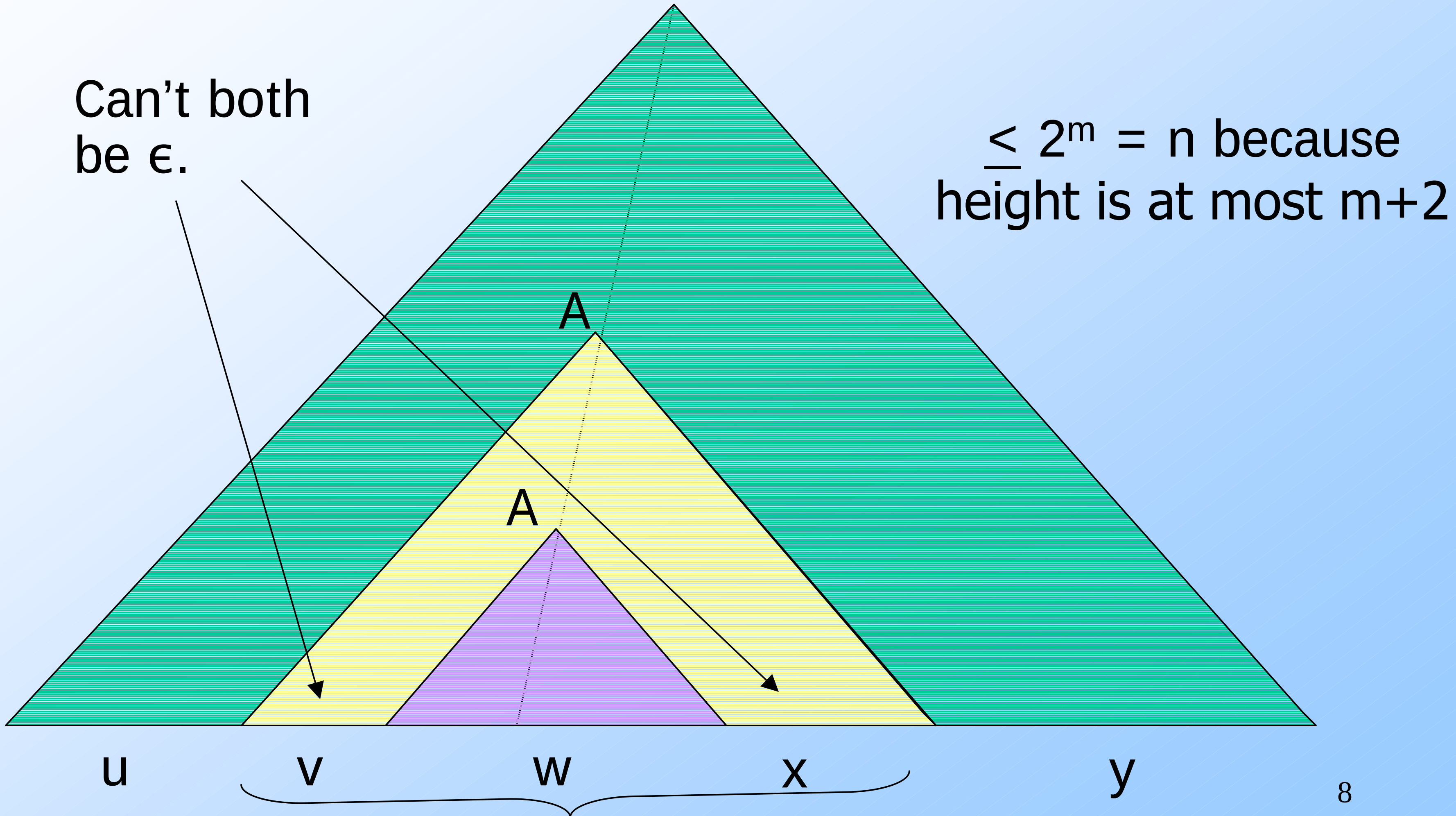
- If all paths in the parse tree of a CNF grammar are of length $\leq$ m+1, then the longest yield has length 2^{m-1}, as in:



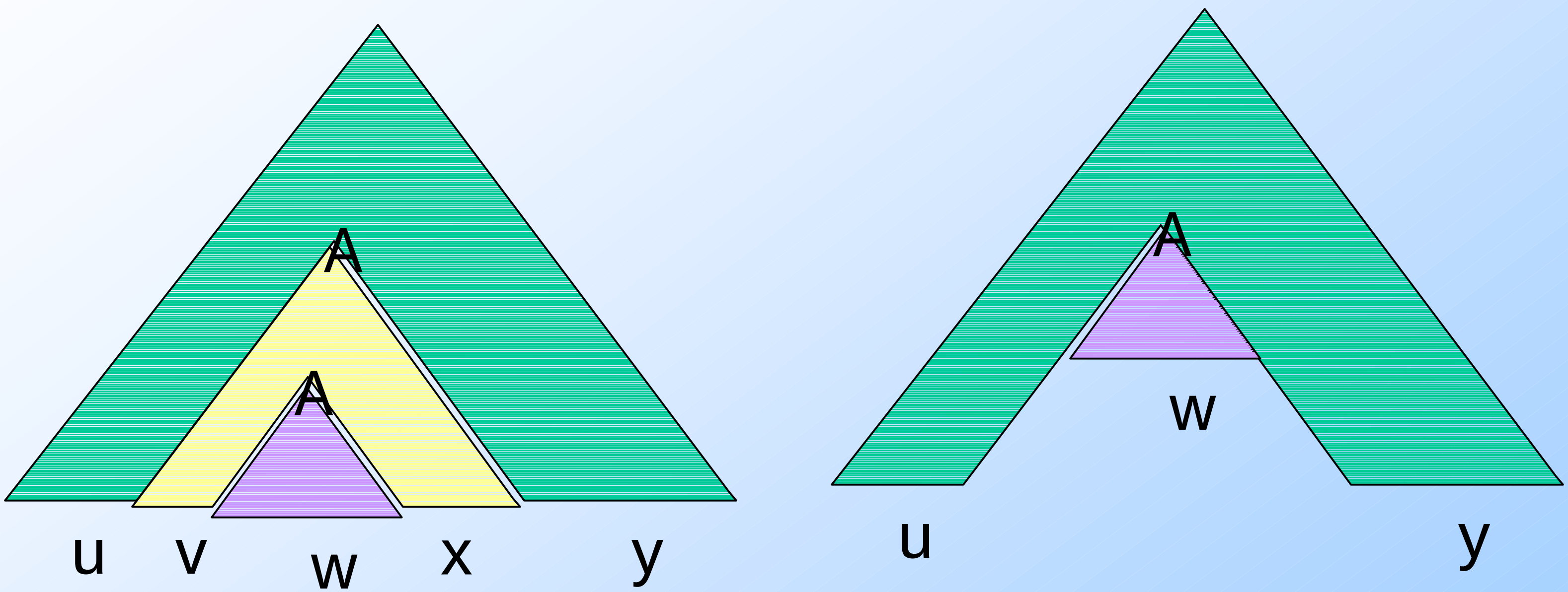
Back to the Proof of the Pumping Lemma

- Now we know that the parse tree for z has a path with at least m+1 variables.
- Consider some longest path.
- There are only m different variables, so among the lowest m+1 we can find two nodes with the same label, say A.
- The parse tree thus looks like:

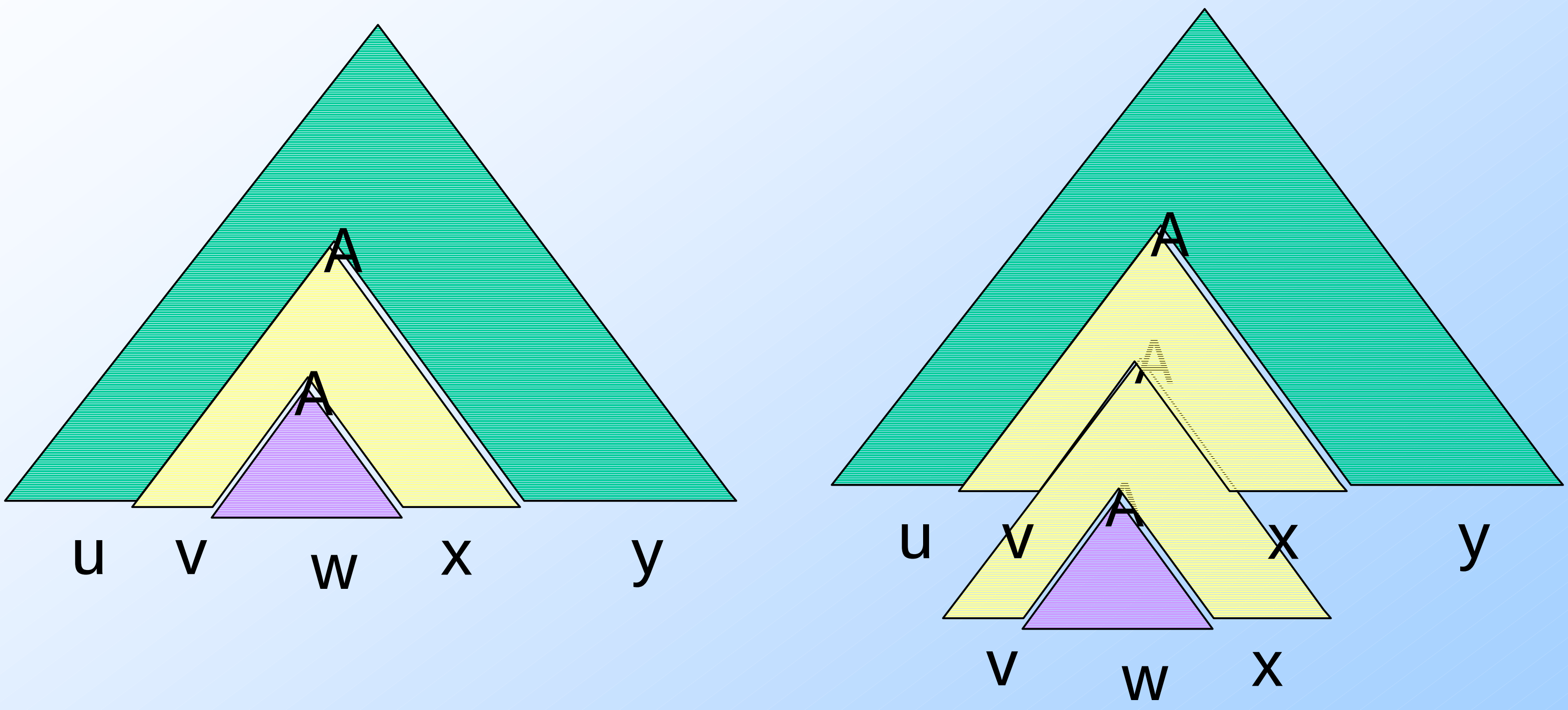
Parse Tree in the Pumping-Lemma Proof



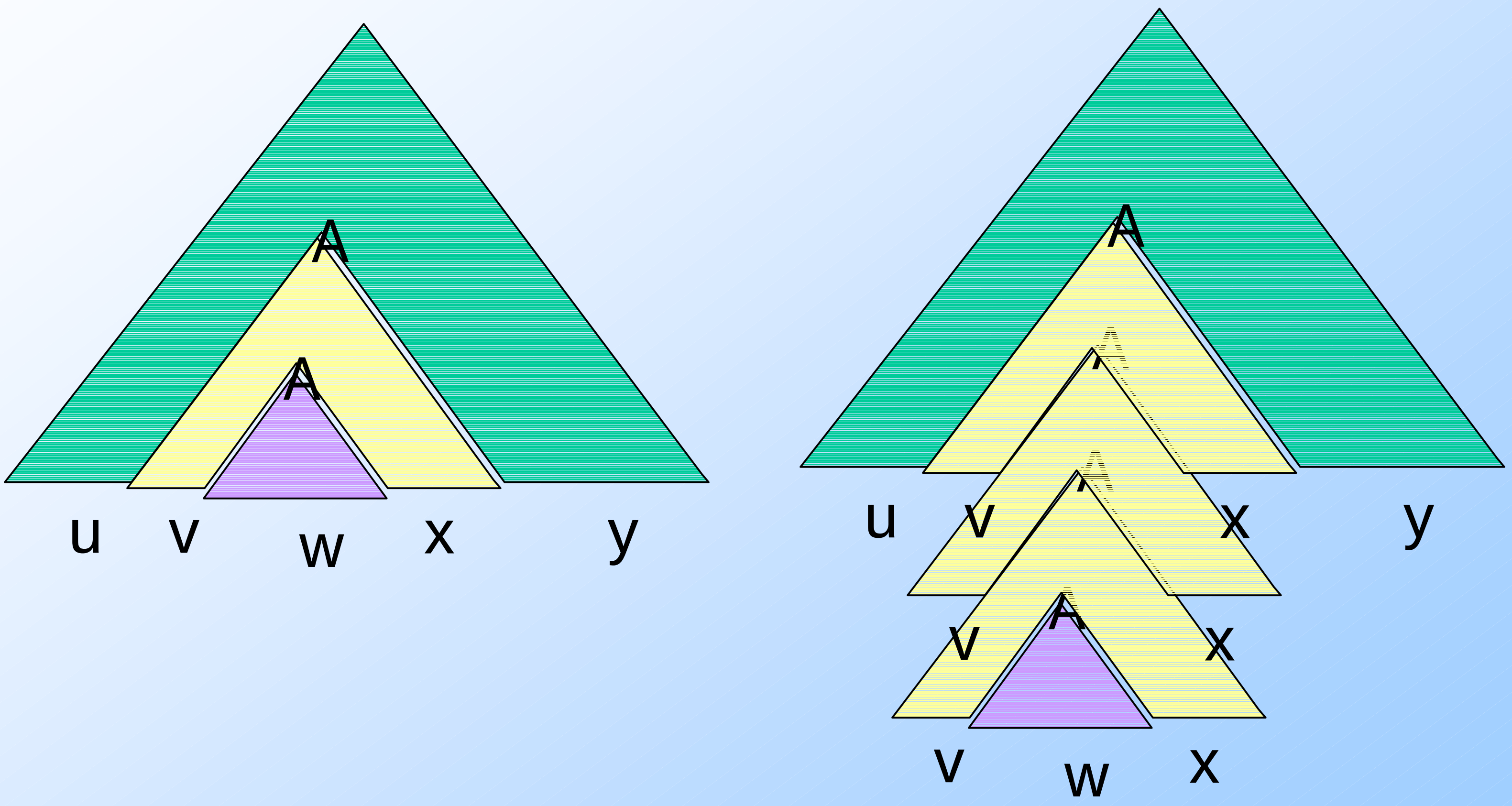
Pump Zero Times



Pump Twice



Pump Thrice Etc., Etc.



Using the Pumping Lemma

- Non-CFL's typically involve trying to match two pairs of counts or match two strings.
- Example: The text uses the pumping lemma to show that {ww | w in **(0+1)***} is not a CFL.

Example 1

$A = \{0^m 1^m 2^m \mid m \geq 0\}$ is not context free.

Proof Assume, to the contrary, that A is context free. By Pumping Lemma there exists a constant n such that every $z \in A$ of length $\geq n$ is divided into $z = uvwxy$ such that $|vwx| \leq n$, $|vx| \geq 1$, and for every $i \geq 0$, $uv^iwx^iy \in A$.

Let $z = 0^n 1^n 2^n$. Since $|vwx| \leq n$, vwx is either in 0^*1^* or 1^*2^*. So it is not the case uv^2wx^2y has the same number of 0s, 1s, as 2s. ∎

A better way to write the proof:

Proof. Pick z = 0^n 1^n 2^n. Consider partition uvwxy = z such that |vwx| <= n and |vx| > 0.

0 ... 0 1 ... 1 2 ... 2

vwx

vwx

Case 1: vwx doesn't contain any 2. Then, u v^2 w x^2 y has more 0s or 1s than 2s.

Case 2: vwx contains a 2 (and thus doesn't contain any 0). Then, u v^2 w x^2 y has more 1s or 2s than 0s.

In either case, pumping doesn't work.

Compare with { 0^a # 0^b # 0^{a+b} | a,b >= 0}, which is a CFL.

Example 2

$B = \{a\#b\#c \mid a, b$ and c are binary numbers such that $a + b = c\}$ is not context free.

Proof Assume, to the contrary, that B is context free. Let *n* be the constant from Pumping Lemma for B. Let z $= 10^{n}\#10^{n}\#10^{n+1}$, where $a = b = 2^{n}$ and $c = 2^{n+1}$. Let uvwxy be the decomposition of z as in the lemma with |vwx| <= n and |vx| > 0.

For "pumping" to be possible, v has to be a nonempty part of a or that of b and x a nonempty part of c. If v either is a part of a or contains the '1' of b, since $|\text{vwx}| \leq n$, x cannot contain a part of c. Thus, v is a part of b and v $\in 0^{*}$.

1 0 ... 0 # 1 0 ... 0 # 1 0 ... 0 0

v w x

Proof Continued

If x contains the first symbol of c, then uwy is not in B because now c is 0 while $a = 2^n$.

If $x \in 0^*$, then $uv^2wx^2y \notin B$ because now the equation becomes $2^n + 2^m = 2^r$ for some $m > n$.

Thus, B is not context-free. ∎

Example 3

$C = \{ww \mid w \in \{0,1\}^*\}$ is not context free.

Proof Assume C is context free. Let n the constant from the pumping lemma for C.

0 ... 0 1 ... 1 0 ... 0 1 ... 1

v w x

v w x

v w x

Let $z = 0^n 1^n 0^n 1^n$, which is in C.

Let $z = uvwxy$ be the decomposition of z such that $|vx| > 0$, $|vwx| \leq n$, and for every $i \geq 0$, $uv^i wx^i y \in C$.

If v contains a symbol from the first 0^n then x cannot contain one from the second 0^n, so pumping doesn't work. If v contains only symbols from the first 1^n then x cannot contain one from the second 1^n, so pumping doesn't work. If v contains only symbols from the second $0^n 1^n$ then pumping does not work. ∎

In all three cases, u w y would not be in C!

Application

Corollary. *The class of context-free languages is not closed under intersection.*

Proof Let $L_1 = \{0^i 1^j 2^k \mid i = j\}$ and $L_2 = \{0^i 1^j 2^k \mid j = k\}$. Then L_1 and L_2 are both context-free. If the class were closed under intersection then $L_1 \cap L_2 = \{0^m 1^m 2^m \mid m \geq 0\}$ were context-free. ∎

Corollary. *The class of context-free languages is not closed under complement.*

Closure Properties of CFL's

Consider a mapping

$$s : \Sigma \rightarrow 2^{\Delta^*}$$

In other words, we map a letter of Σ to a language over Δ

where Σ and Δ are finite alphabets. Let $w \in \Sigma^*$, where $w = a_1 a_2 \cdots a_n$, and define

$$s(a_1 a_2 \cdots a_n) = s(a_1) \cdot s(a_2) \cdot \cdots \cdot s(a_n)$$

and, for $L \subseteq \Sigma^*$,

$$s(L) = \bigcup_{w \in L} s(w)$$

Such a mapping s is called a *substitution*.

Example: $\Sigma = \{0, 1\}, \Delta = \{a, b\}$,
$s(0) = \{a^n b^n : n \geq 1\}, s(1) = \{aa, bb\}$.

Let $w = 01$. Then $s(w) = s(0) \cdot s(1) =$
$\{a^n b^n aa : n \geq 1\} \cup \{a^n b^{n+2} : n \geq 1\}$

Let $L = \{0\}^*$. Then $s(L) = (s(0))^* =$
$\{a^{n_1} b^{n_1} a^{n_2} b^{n_2} \cdots a^{n_k} b^{n_k} : k \geq 0, n_i \geq 1\}$

Theorem 7.23: Let L be a CFL over Σ, and s a substitution, such that $s(a)$ is a CFL, $\forall a \in \Sigma$. Then $s(L)$ is a CFL.

We start with grammars

$$G = (V, \Sigma, P, S)$$

e.g., S -> 0S | ε

for L, and

$$G_a = (V_a, T_a, P_a, S_a)$$

e.g., X -> aXb | ab

for each $s(a)$. We then construct

$$G' = (V', T', P', S\ )$$

where

e.g., S -> XS | ε
X -> aXb | ab

$V' = (\bigcup_{a \in \Sigma} V_a) \cup V$

$T' = \bigcup_{a \in \Sigma} T_a$

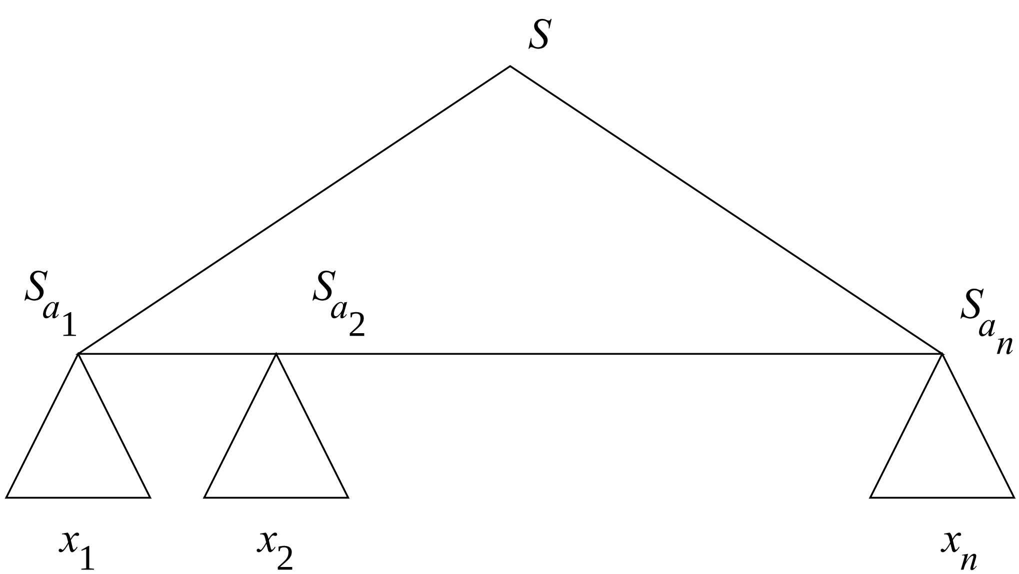
$P' = \bigcup_{a \in \Sigma} P_a$ plus the productions of P with each a in a body replaced with symbol S_a.

Now we have to show that

- $L(G') = s(L)$.

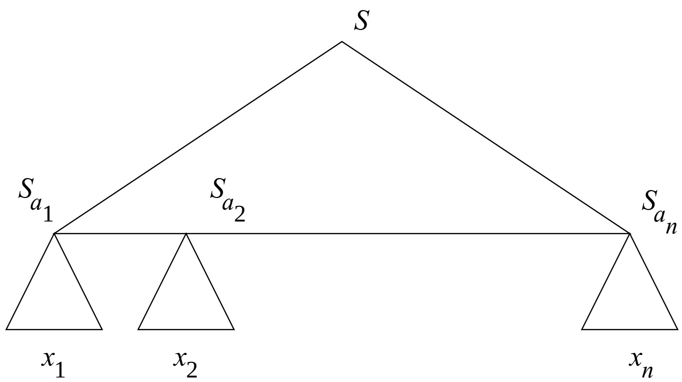
Let $w \in s(L)$. Then $\exists x = a_1 a_2 \cdots a_n$ in L, and $\exists x_i \in s(a_i)$, such that $w = x_1 x_2 \cdots x_n$.

A derivation tree in G' will look like



Thus we can generate $S_{a_1} S_{a_2} \cdots S_{a_n}$ in G' and from there we generate $x_1 x_2 \cdots x_n = w$. Thus $w \in L(G')$.

Then let $w \in L(G')$. Then the parse tree for w must again look like



Now delete the dangling subtrees. Then you have yield

$$S_{a_1} S_{a_2} \cdots S_{a_n}$$

where $a_1 a_2 \cdots a_n \in L(G)$. Now w belongs to $s(a_1 a_2 \cdots a_n)$, which is contained in $S(L)$.

Applications of the Substitution Theorem

Theorem 7.24: The CFL's are closed under (i) : union, (ii) : concatenation, (iii) : Kleene closure and positive closure $+$, and (iv) : homomorphism.

Proof: (i): Let L_1 and L_2 be CFL's, let $L = \{1, 2\}$, and $s(1) = L_1, s(2) = L_2$.
Then $L_1 \cup L_2 = s(L)$.

(ii) : Here we choose $L = \{12\}$ and s as before.
Then $L_1 {\cdot} L_2 = s(L)$

(iii) : Suppose L_1 is CF. Let $L = \{1\}^*, s(1) = L_1$. Now $L_1^* = s(L)$. Similar proof for $+$.

(iv) : Let L_1 be a CFL over Σ, and h a homomorphism on Σ. Then define s by

$$a \mapsto \{h(a)\}$$

Then $h(L_1) = s(L_1)$.

Theorem: If L is CF, then so is L^R.

Proof: Suppose L is generated by $G = (V, T, P, S)$. Construct $G^R = (V, T, P^R, S)$, where

$$P^R = \{A \to \alpha^R : A \to \alpha \in P\}$$

Show at home by inductions on the lengths of the derivations in G (for one direction) and in G^R (for the other direction) that $(L(G))^R = L(G^R)$.

E.g., $\{0^{m}1^{n}2^{m+n}\}^R = \{2^{m+n}1^{m}0^{n}\}$

Let $L_1 = \{0^n 1^n 2^i : n \geq 1, i \geq 1\}$. The L_1 is CF with grammar

$$S \rightarrow AB$$
$$A \rightarrow 0A1|01$$
$$B \rightarrow 2B|2$$

Also, $L_2 = \{0^i 1^n 2^n : n \geq 1, i \geq 1\}$ is CF with grammar

$$S \rightarrow AB$$
$$A \rightarrow 0A|0$$
$$B \rightarrow 1B2|12$$

However, $L_1 \cap L_2 = \{0^n 1^n 2^n : n \geq 1\}$ which is not CF (as shown in the last lecture).

E.g., {palindromes} $\cap$ {even length strings} = {even length palindromes}

Theorem 7.27: If L is CF, and R regular, then $L \cap R$ is CF.

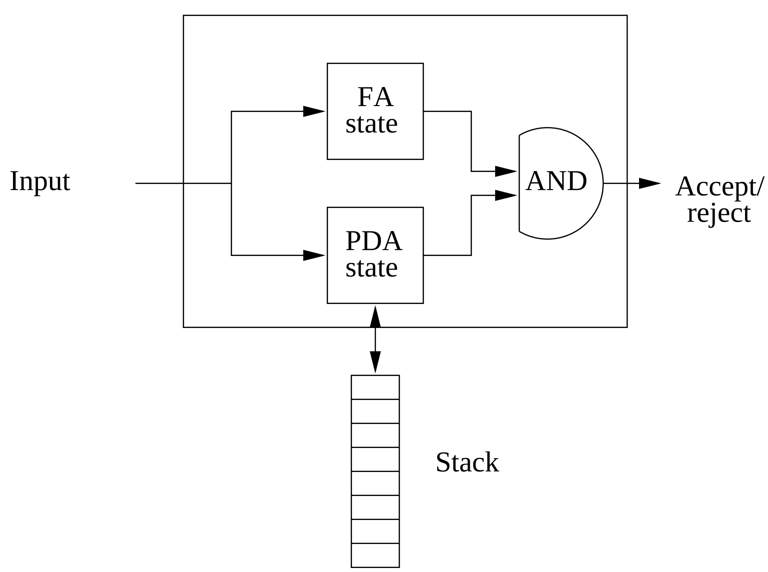
Proof: Let L be accepted by PDA

$$P = (Q_P, \Sigma, \Gamma, \delta_P, q_P, Z_0, F_P)$$

by final state, and let R be accepted by DFA

$$A = (Q_A, \Sigma, \delta_A, q_A, F_A)$$

We'll construct a PDA for $L \cap R$ according to the picture



Formally, define

$$P' = (Q_P \times Q_A, , \Sigma, \Gamma, \delta, (q_P, q_A), Z_0, F_P \times F_A)$$

where

$$\delta((q,p),a,X) = \{((r, \hat{\delta}_A(p,a)), \gamma) : (r,\gamma) \in \delta_P(q,a,X)\}$$

where a is in $\Sigma \cup \{\varepsilon\}$

Prove at home by an induction $\overset{*}{\vdash}$, both for P and for P' that

$$(q_P, w, Z_0) \overset{*}{\vdash} (q, \epsilon, \gamma) \text{ in } P$$

if and only if

$$((q_P, q_A), w, Z_0) \overset{*}{\vdash} \Big((q, \hat{\delta}_A(q_A, w)), \epsilon, \gamma\Big) \text{ in } P'$$

The claim then follows (Why?)

Theorem 7.29: Let L, L_1, L_2 be CFL's and R regular. Then

1. $L \setminus R$ is CF

E.g., Dyck language \ [()]* = nested balanced parentheses

2. $\bar{L}$ is not necessarily CF

3. $L_1 \setminus L_2$ is not necessarily CF

Proof:

1. $\bar{R}$ is regular, $L \cap \bar{R}$ is CF, and $L \cap \bar{R} = L \setminus R$.

2. If $\bar{L}$ always was CF, it would follow that

$$L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$$

always would be CF.

An example?

Non-squares!

3. Note that Σ^* is CF, so if $L_1 \setminus L_2$ was always CF, then so would $\Sigma^* \setminus L = \bar{L}$.

Inverse homomorphism

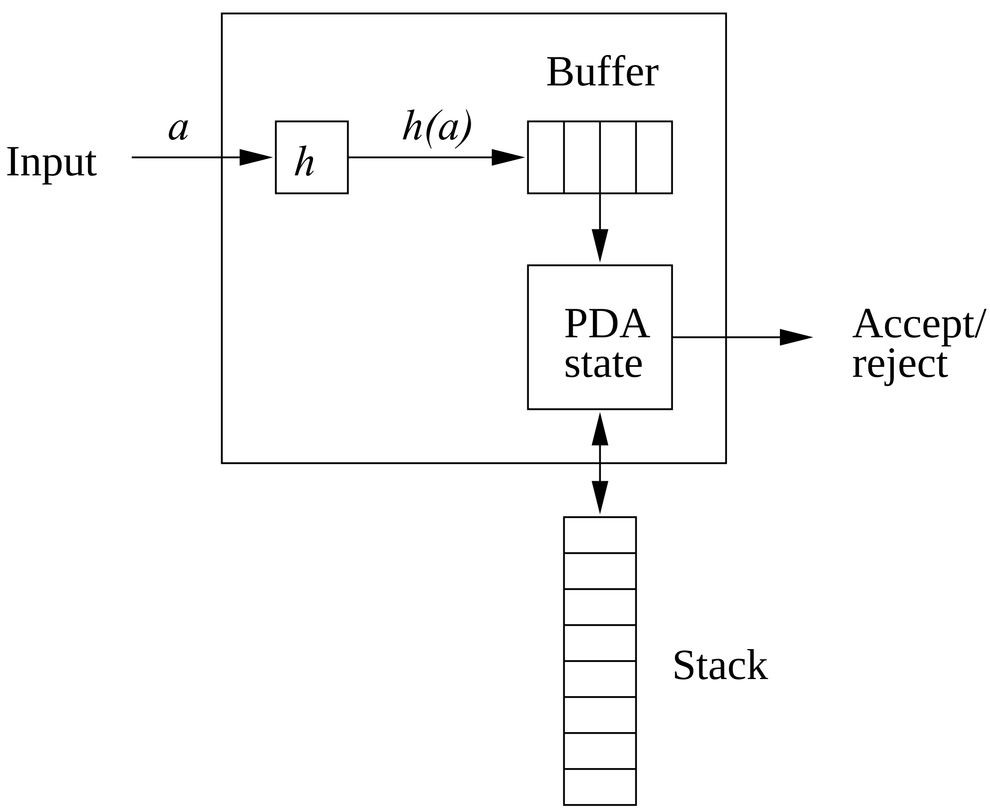
Let $h : \Sigma \to \Theta^*$ be a homom. Let $L \subseteq \Theta^*$, and define

$$h^{-1}(L) = \{w \in \Sigma^* : h(w) \in L\}$$

Now we have

Theorem 7.30: Let L be a CFL, and h a homomorphism. Then $h^{-1}(L)$ is a CFL.

Proof: The plan of the proof is



Let L be accepted by PDA

$$P = (Q, \Theta, \Gamma, \delta, q_0, Z_0, F)$$

We construct a new PDA

$$P' = (Q', \Sigma, \Gamma, \delta', (q_0, \epsilon), Z_0, F \times \{\epsilon\})$$

where

$$Q' = \{(q, x) : q \in Q, x \in \mathit{suffix}(h(a)), a \in \Sigma\}$$

$$\delta'((q, \epsilon), a, X) = \{((q, h(a)), X) : \epsilon \neq a \in \Sigma, q \in Q, X \in \Gamma\}$$

$$\delta'((q, bx), \epsilon, X) = \{((p, x), \gamma) : (p, \gamma) \in \delta(q, b, X), b \in \Sigma \cup \{\epsilon\}, q \in Q, X \in \Gamma\}$$

Show at home by suitable inductions that

- $(q_0, h(w), Z_0) \overset{*}{\vdash} (p, \epsilon, \gamma)$ in P if and only if $((q_0, \epsilon), w, Z_0) \overset{*}{\vdash} ((p, \epsilon), \epsilon, \gamma)$ in P'.

Note that $h(\varepsilon) = \varepsilon$.

Decision Properties of CFL's

We'll look at the following:

- Complexity of converting among CFG's and PDA 's

- Converting a CFG to CNF

- Testing $L(G) \neq \emptyset$, for a given G

- Testing $w \in L(G)$, for a given w and fixed G.

- Preview of undecidable CFL problems

Converting between CFGs and PDA's

- Input size is n.
- n is the *total* size of the input CFG or PDA.

The following work in time $O(n)$

1. Converting a CFG to a PDA (slide 203)
2. Converting a "final state" PDA to a "null stack" PDA (slide 199)
3. Converting a "null stack" PDA to a "final state" PDA (slide 195)

Avoidable exponential blow-up

For converting a PDA to a CFG we have

(slide 210)

At most n^3 variables of the form $[pXq]$

If $(\mathbf{r}, Y_1 Y_2 \cdots Y_k) \in \delta(\mathbf{q}, a, X)\}$, we'll have $O(n^n)$ rules of the form

$$[\mathbf{q}X r_k] \to a[\mathbf{r}Y_1 r_1] \cdots [r_{k-1} Y_k r_k]$$

- By introducing $k-2$ new states we can modify the PDA to push at most *one* symbol per transition. Illustration on blackboard in class.

Put $(r_{Y3\ldots Yk}, Y_2 Y_1)$ in $\delta(q,a,X)$
Put $(r_{Y4\ldots Yk}, Y_3 Y_2)$ in $\delta(r_{Y3\ldots Yk}, \varepsilon, Y_2)$
...

- Now, k will be ≤ 2 for all rules.

- Total length of all transitions is still $O(n)$.

- Now, each transition generates at most n^2 productions

- Total size (and time to calculate) the grammar is therefore $O(n^3)$.

Converting into CNF

Good news:

1. Computing $r(G)$ and $g(G)$ and eliminating useless symbols takes time $O(n)$. This will be shown shortly

   (slides 229,232,234)

2. Size of $u(G)$ and the resulting grammar with productions P_1 is $O(n^2)$

   (slides 244,245)

3. Arranging that bodies consist of only variables is $O(n)$

   (slide 248)

4. Breaking of bodies is $O(n)$ (slide 248)

Bad news:

- Eliminating the nullable symbols can make the new grammar have size $O(2^n)$

(slide 236)

The bad news are avoidable:

Break bodies first before eliminating nullable symbols

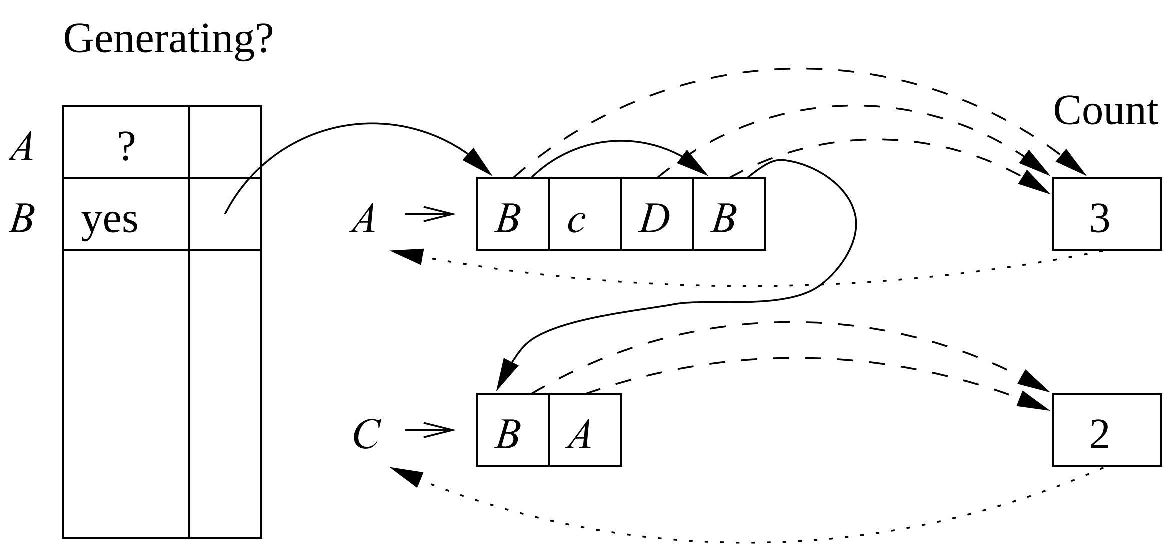
- Conversion into CNF is $O(n^2)$

Testing emptiness of CFL's

$L(G)$ is non-empty if the start symbol S is generating.

A naive implementation on $g(G)$ takes time $O(n^2)$.

$g(G)$ can be computed in time $O(n)$ as follows:



Creation and initialzation of the array is $O(n)$

Creation and initialzation of the links and counts is $O(n)$

When a count goes to zero, we have to

1. Finding the head variable A, checking if it already is "yes" in the array, and if not, queueing it is $O(1)$ per production. Total $O(n)$

2. Following links for A, and decreasing the counters. Takes time $O(n)$.

Total time is $O(n)$.

What if L is given as a PDA?
How to test if L(G) is infinite?

The membership question

$$\boxed{w \in L(G)\textbf{?}}$$

Inefficient way:

Suppose G is CNF, test string is w, with $|w| = n$. Since the parse tree is binary, there are $2n - 1$ internal nodes.

Generate *all* binary parse trees of G with $2n-1$ internal nodes.

Check if any parse tree generates w

CYK-algo for membership testing

The grammar G is fixed and in CNF.

Input is $w = a_1 a_2 \cdots a_n$

We construct a triangular table, where X_{ij} contains all variables A, such that

$$A \overset{*}{\underset{G}{\Rightarrow}} a_i a_{i+1} \cdots a_j$$

$$\begin{array}{|lllll}
X_{15} & & & & \\
X_{14} & X_{25} & & & \\
X_{13} & X_{24} & X_{35} & & \\
X_{12} & X_{23} & X_{34} & X_{45} & \\
X_{11} & X_{22} & X_{33} & X_{44} & X_{55} \\
\hline
\multicolumn{1}{l}{a_1} & a_2 & a_3 & a_4 & a_5
\end{array}$$

To fill the table we work row-by-row, upwards

The first row is computed in the basis, the subsequent ones in the induction.

Basis: $X_{ii} == \{A : A \to a_i \text{ is in } G\}$

Induction:

We wish to compute X_{ij}, which is in row $j - i + 1$.

$A \in X_{ij}$, if
$A \stackrel{*}{\Rightarrow} a_i a_i + 1 \cdots a_j$, if
for some $k < j$, and $A \to BC$, we have
$B \stackrel{*}{\Rightarrow} a_i a_{i+1} \cdots a_k$, and $C \stackrel{*}{\Rightarrow} a_{k+1} a_{k+2} \cdots a_j$, if
$B \in X_{ik}$, and $C \in X_{(k+1)j}$

Example:

G has productions

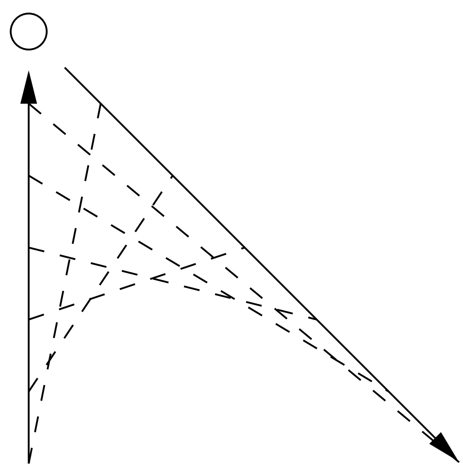
$$
\begin{aligned}
S &\rightarrow AB|BC \\
A &\rightarrow BA|a \\
B &\rightarrow CC|b \\
C &\rightarrow AB|a
\end{aligned}
$$

$\{S,A,C\}$				
-	$\{S,A,C\}$			
-	$\{B\}$	$\{B\}$		
$\{S,A\}$	$\{B\}$	$\{S,C\}$	$\{S,A\}$	
$\{B\}$	$\{A,C\}$	$\{A,C\}$	$\{B\}$	$\{A,C\}$
b	a	a	b	a

To compute X_{ij} we need to compare at most n pairs of previously computed sets:

$$(X_{ii}, X_{i+1,j}), (X_{i,i+1}, X_{i+2,j}), \ldots, (X_{i,j-1}, X_{jj})$$

as suggested below



For $w = a_1 \cdots a_n$, there are $O(n^2)$ entries X_{ij} to compute.

For each X_{ij} we need to compare at most n pairs $(X_{ik}, X_{k+1,j})$.

Total work is $O(n^3)$.

Preview of undecidable CFL problems

The following are undecidable:

1. Is a given CFG G ambiguous?

2. Is a given CFL inherently ambiguous?

3. Is the intersection of two CFL's empty?

4. Are two CFL's the same?

5. Is a given CFL universal (equal to Σ^*)?

Open: Does a DFA accept any prime number?

∞

Undecidability

Everything is an Integer

Countable and Uncountable Sets

Turing Machines

Recursive and Recursively Enumerable Languages

Integers, Strings, and Other Things

- Data types have become very important as a programming tool.
- But at another level, there is only one type, which you may think of as integers or strings.

Example: Text

- Strings of ASCII or Unicode characters can be thought of as binary strings, with 8 or 16 bits/character.
- Binary strings can be thought of as integers.
- It thus makes sense to talk about “the i-th string”.

Binary Strings to Integers

- There's a small glitch:
  - If you think them simply as binary integers, then strings like 101, 0101, 00101, ... all appear to represent 5.
- Fix by prepending a "1" to the string before converting to an integer.
  - Thus, 101, 0101, and 00101 are the 13^{th}, 21^{st}, and 37^{th} strings, respectively.

Example: Images

- Represent an image in (say) GIF.
- The GIF file is an ASCII string.
- Convert string to binary.
- Convert binary string to integer.
- Now we have a notion of “the i-th image”.

Example: Proofs

- A formal proof is a sequence of logical expressions, each of which follows from the ones before it.
- Encode mathematical expressions of any kind in Unicode.
- Convert expression to a binary string and then an integer.

Proofs – (2)

- But since a proof is a sequence of expressions, it would be convenient to have a simple way to separate them.
- Also, we need to indicate which expressions are given.

Proofs – (3)

- Quick-and-dirty way to introduce new symbols into binary strings:
  1. Given a binary string, precede each bit by 0.
     - Example: 101 becomes 010001.
  2. Use strings of two or more 1's as the special symbols.
     - Example: 111 = "the following expression is given"; 11 = "end of expression."

Example: Encoding Proofs

111010001111111000000101110101…

A given expression follows

An ex-pression

Notice this 1 could not be part of the "end"

End of expression

A given expression follows

Expression

End

Example: Programs

- Programs are just another kind of data.
- Represent a program in ASCII.
- Convert to a binary string, then to an integer.
- Thus, it makes sense to talk about "the i-th program".
- Hmm...There aren't all that many programs. Each (decision) program accepts one language.

Finite Sets

- Intuitively, a *finite set* is a set for which there is a particular integer that is the count of the number of members.
- Example: {a, b, c} is a finite set; its *cardinality* is 3.
- It is impossible to find a 1-1 mapping between a finite set and a proper subset of itself.

Infinite Sets

- Formally, an *infinite set* is a set for which there is a 1-1 correspondence between itself and a proper subset of itself.
- Example: the positive integers {1, 2, 3, ...} is an infinite set.
  - There is a 1-1 correspondence 1<->2, 2<->4, 3<->6,... between this set and a proper subset (the set of even integers).

Countable Sets

- A *countable set* is a set with a 1-1 correspondence with the positive integers.
  - Hence, all countable sets are infinite.
- Example: All integers.
  - 0<->1; -i <-> 2i; +i <-> 2i+1.
  - Thus, order is 0, -1, 1, -2, 2, -3, 3,...
- Examples: set of binary strings, set of Java programs.

Example: Pairs of Integers

- Order the pairs of positive integers first by sum, then by first component:
- [1,1], [2,1], [1,2], [3,1], [2,2], [1,3], [4,1], [3,2],..., [1,4], [5,1],...
- Interesting exercise: Figure out the function f(i,j) such that the pair [i,j] corresponds to the integer f(i,j) in this order.

Enumerations

- An *enumeration* of a set is a 1-1 correspondence between the set and the positive integers.
- Thus, we have seen enumerations for strings, programs, proofs, and pairs of integers.

How Many Languages?

- Are the languages over {0,1}* countable?
- No; here's a proof.
- Suppose we could enumerate all languages over {0,1}* and talk about "the i-th language."
- Consider the language L = { w | w is the i-th binary string and w is not in the i-th language}.

Proof – Continued

L_j

- Clearly, L is a language over {0,1}*.
- Thus, it is the j-th language for some particular j.
- Let x be the j-th string.
- Is x in L?
  - If so, x is not in L by definition of L.
  - If not, then x is in L by definition of L.

x

Recall: L = { w | w is the i-th binary string and w is not in the i-th language}.

j-th

Diagonalization Picture

Strings

Languages

	1	2	3	4	5	...
1	1	0	1	1	0	...
2		1				
3			0			
4				0		
5					1	
...						...

Diagonalization Picture

Strings

Flip each diagonal entry

Languages

	1	2	3	4	5	...
1	0	0	1	1	0	...
2		0				
3			1			
4				1		
5					0	
...						...

Can’t be a row – it disagrees in an entry of each row.

Proof – Concluded

- We have a contradiction: x is neither in L nor not in L, so our sole assumption (that there was an enumeration of the languages) is wrong.
- Comment: This is really bad; there are more languages than programs.
- E.g., there are languages that are not accepted by any program/algorithm.

Recall languages are essentially decision problems and algorithms accepting the languages basically solve the decision problems.

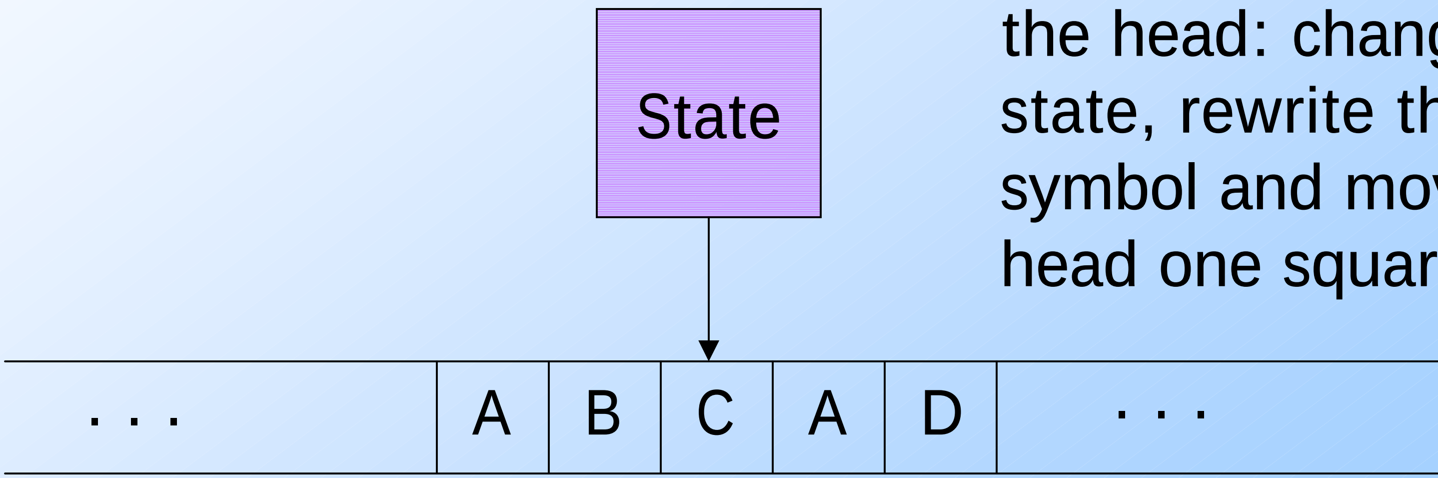
Hungarian Arguments

- We have shown the existence of a language with no algorithm to test for membership, but we have no way to exhibit a particular language with that property.
- A proof by counting the things that work and claiming they are fewer than all things is called a *Hungarian argument*.

Turing-Machine Theory

- The purpose of the theory of Turing machines is to prove that certain specific languages have no algorithm.
- Start with a language about Turing machines themselves.
- Reductions are used to prove more common questions undecidable.

Picture of a Turing Machine



Action: based on the state and the tape symbol under the head: change state, rewrite the symbol and move the head one square.

Infinite tape with squares containing tape symbols chosen from a finite alphabet

Why Turing Machines?

- Why not deal with C programs or something like that?
- Answer: You can, but it is easier to prove things about TM's, because they are so simple.
  - And yet they are as powerful as any computer.
    - More so, in fact, since they have infinite memory.

Then Why Not Finite-State Machines to Model Computers?

- In principle, you could, but it is not instructive.
- Programming models don't build in a limit on memory.
- In practice, you can go to Fry's and buy another disk.
- But finite automata vital at the chip level (model-checking).

Turing-Machine Formalism

- A TM is described by:
  1. A finite set of *states* (Q, typically).
  2. An *input alphabet* (Σ, typically).
  3. A *tape alphabet* (Γ, typically; contains Σ).
  4. A *transition function* (δ, typically).
  5. A *start state* (q_0, in Q, typically).
  6. A *blank symbol* (B, in Γ− Σ, typically).
     - All tape except for the input is blank initially.
  7. A set of *final states* (F ⊆ Q, typically).

Conventions

- a, b, ... are input symbols.
- ..., X, Y, Z are tape symbols.
- ..., w, x, y, z are strings of input symbols.
- α, β,... are strings of tape symbols.

The Transition Function

- Takes two arguments:
  1. A state, in Q.
  2. A tape symbol in Γ.
- δ(q, Z) is either undefined or a triple of the form (p, Y, D).
  - p is a state.
  - Y is the new tape symbol.
  - D is a *direction*, L or R.

Actions of the TM

- If $\delta(q, Z) = (p, Y, D)$ then, in state q, scanning Z under its tape head, the TM:
  1. Changes the state to p.
  2. Replaces Z by Y on the tape.
  3. Moves the head one square in direction D.
     - D = L: move left; D = R; move right.

Example: Turing Machine

- This TM scans its input right, looking for a 1.
- If it finds one, it changes it to a 0, goes to final state f, and halts.
- If it reaches a blank, it changes it to a 1 and moves left.

Example: Turing Machine – (2)

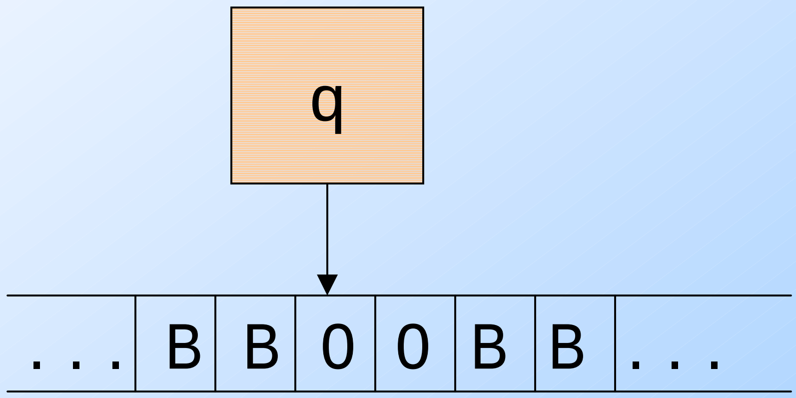
- States = {q (start), f (final)}.
- Input symbols = {0, 1}.
- Tape symbols = {0, 1, B}.
- $\delta(q, 0) = (q, 0, R)$.
- $\delta(q, 1) = (f, 0, R)$.
- $\delta(q, B) = (q, 1, L)$.

Simulation of TM

$\delta(q, 0) = (q, 0, R)$

$\delta(q, 1) = (f, 0, R)$

$\delta(q, B) = (q, 1, L)$



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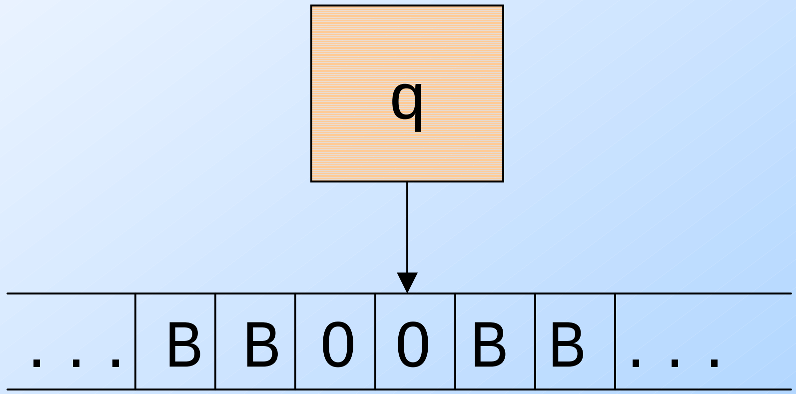

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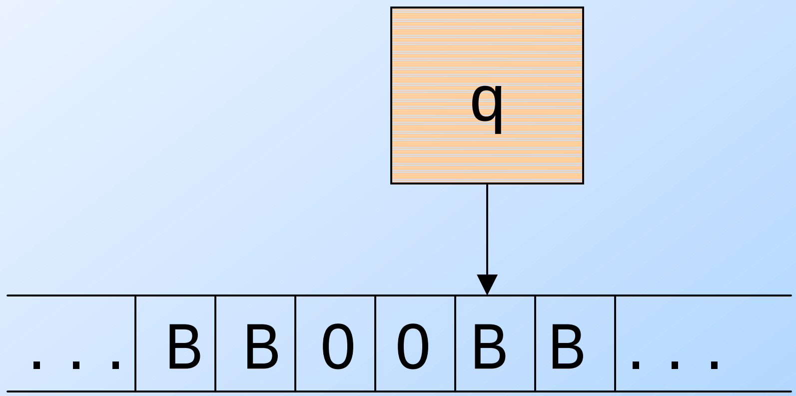

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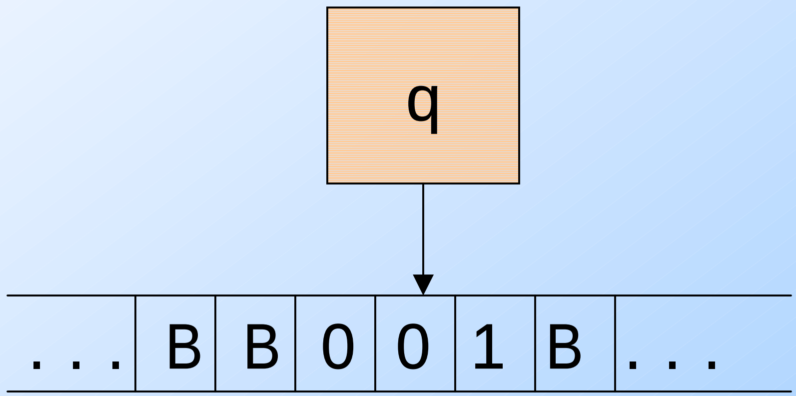

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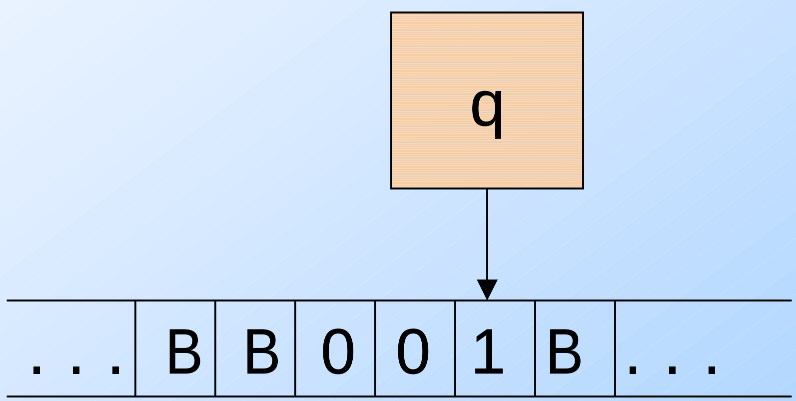

Simulation of TM

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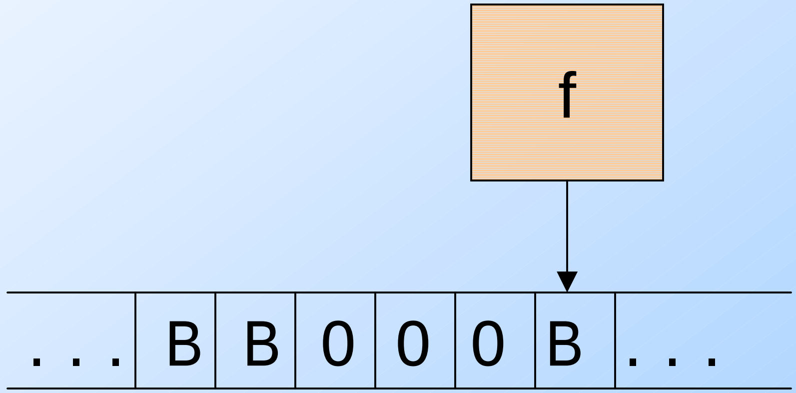

No move is possible.
The TM halts and
accepts.

Instantaneous Descriptions of a Turing Machine

- Initially, a TM has a tape consisting of a string of input symbols surrounded by an infinity of blanks in both directions.
- The TM is in the start state, and the head is at the leftmost input symbol.

TM ID's – (2)

- An ID is a string $\alpha q \beta$, where $\alpha\beta$ is the tape between the leftmost and rightmost nonblanks (inclusive).
- The state q is immediately to the left of the tape symbol scanned.
- If q is at the right end, it is scanning B.
  - If q is scanning a B at the left end, then consecutive B's at and to the right of q are part of β.

TM ID's – (3)

- As for PDA's we may use symbols $\vdash$ and $\vdash^*$ to represent "becomes in one move" and "becomes in zero or more moves," respectively, on ID's.
- Example: The moves of the previous TM are $q00 \vdash 0q0 \vdash 00q \vdash 0q01 \vdash 00q1 \vdash 000f$

Formal Definition of Moves

1. If $\delta(q, Z) = (p, Y, R)$, then
   - $\alpha qZ\beta \vdash \alpha Yp\beta$
   - If Z is the blank B, then also $\alpha q \vdash \alpha Yp$
2. If $\delta(q, Z) = (p, Y, L)$, then
   - For any X, $\alpha XqZ\beta \vdash \alpha pXY\beta$
   - In addition, $qZ\beta \vdash pBY\beta$

Languages of a TM

- A TM defines a language by final state, as usual.
- $L(M) = \{w \mid q_0w\vdash^* I$, where I is an ID with a final state$\}$.
- Or, a TM can accept a language by halting.
- $H(M) = \{w \mid q_0w\vdash^* I$, and there is no move possible from ID $I\}$.

Equivalence of Accepting and Halting

1. If L = L(M), then there is a TM M’ such that L = H(M’).
2. If L = H(M), then there is a TM M” such that L = L(M”).

Proof of 1: Acceptance -> Halting

- Modify M to become M' as follows:
  1. For each final state of M, remove any moves, so M' halts in that state.
  2. Avoid having M' accidentally halt.
     - Introduce a new state s, which runs to the right forever; that is $\delta(s, X) = (s, X, R)$ for all symbols X.
     - If q is not final, and $\delta(q, X)$ is undefined, let $\delta(q, X) = (s, X, R)$.

Proof of 2: Halting -> Acceptance

- Modify M to become M" as follows:
  1. Introduce a new state f, the only final state of M".
  2. f has no moves.
  3. If $\delta(q, X)$ is undefined for any state q and symbol X, define it by $\delta(q, X) = (f, X, R)$.

Recursively Enumerable Languages

- We now see that the classes of languages defined by TM's using final state and halting are the same.
- This class of languages is called the *recursively enumerable languages*.
  - Why? The term actually predates the Turing machine and refers to another notion of computation of functions.

AMB = {<G> | G is an ambiguous CFG}

Recursive Languages

- An *algorithm* is a TM that is guaranteed to halt whether or not it accepts.
- If L = L(M) for some TM M that is an algorithm, we say L is a *recursive* (or decidable) language.
  - Why? Again, don't ask; it is a term with a history.

Church-Turing Thesis: Halting Turing machines are equivalent to intuitive notion of algorithms.

Example: Recursive Languages

- Every CFL is a recursive language.
  - Use the CYK algorithm.
- Every regular language is a CFL (think of its DFA as a PDA that ignores its stack); therefore every regular language is recursive.
- Almost anything you can think of is recursive.

But not HALT = {<M> | M is a TM that halts on every input}
or AMB = {<G> | G is an ambiguous CFG}
or EQCFG = $\{<G_1,G_2> \mid G_1$ and G_2 are CFGs, $L(G_1) = L(G_2)\}$

An example non-recursive (undecidable) language:

A_{TM} = { <M,w> | TM M accepts string w }

Proof. Suppose that A_{TM} is recursive and decided by an algorithm (TM) H. Construct a TM D as follows:

For any input <M> where M is a TM, run H on <M,<M>>, and accept iff H rejects. In other words, D accepts <M> iff M does not accept <M>.

What would D do on <D>?

It should accept <D> iff D rejects <D> !