CS150 Homework 3

Solution keys, spring, 2025

Problem 1. (10 points)

Prove that the following are not regular languages.

(a) $\{0^n1^m2^n \mid n \text{ and } m \text{ are arbitrary integers}\}$.

(b) $\{0^{2n}1^n \mid n \geq 1\}$

(a) **Proof.** Assuming the language L is regular, let p be the pumping-lemma constant. Pick $w = 0^p12^p$. Then when we write $w = xyz$, we know that $|xy| \leq p$, and therefore y consists of only 0's. Thus, xz, which must be in L if L is regular, consists of fewer than p 0's, followed by a 1 and exactly p 2's. That string is not in L, so we contradict the assumption that L is regular. □

(b) **Proof.** Assuming the language L is regular, let p be the pumping-lemma constant. Pick $w= 0^{2p}1^p$. Then when we write $w = xyz$, we know that $|xy| \leq p$, and therefore y consists of only 0's. Thus, $xyyz$, which must be in L if L is regular, consists of more than $2p$ 0's, followed by exactly p 1's. That string is not in L, so we contradict the assumption that L is regular. □

Problem 2. (10 points)

Prove that the following are not regular languages:

The set of strings of 0's and 1's that are of the form ww, that is, same string repeated.

Proof. Assuming the language L is regular, let p be the pumping-lemma constant. Pick a string $0^p1^p0^p1^p$. Then when we write it as xyz, we know that $|xy| \leq p$, and therefore y consists of only 0's. Thus, xz, which must be in L if L is regular, consists of fewer than p 0's, followed by exactly p 1's, then exactly p 0's, and another p 1's. Clearly this string is not of the form ww, so we contradict the assumption that L is regular. $\square$

Problem 3. (Exercise 4.2.3, 10 points)

If L is a language, and a is a symbol, then $a \setminus L$ is the set of string w such that aw is in L. For example, if $L = \{a, aab, baa\}$, then $a \setminus L = \{\epsilon, ab\}$. Prove that if L is regular, so is $a \setminus L$. *Hint:* Remember that the regular languages are closed under reversal and under the quotient operation of Exercise 4.2.2.

Proof. If L is regular, so is L^R (the regular languages are closed under reversal). According to Exercise 4.2.2, we know L^R/a is also regular. Since it is easy to prove $a \setminus L = (L^R/a)^R$, we conclude that $a \setminus L$ is regular.

$\square$

Problem 4. (10 points)

Give an algorithm to tell whether a regular language L contains at least 100 strings.

Algorithm 1 NumberOfStrings(D, n)

Input: D: a black box that tests if a string is in L. n: pumping lemma constant.
Output: Return “yes” if L contains at least 100 strings, otherwise return “no”

```
for i ← n to 2n − 1 do
    for all string w of length i do
        if D(w) = accept then return
        “yes”   // the language is infinite //
count ← 0
for i ← 0 to n − 1 do
    for all string w of length i do
        if D(w) = accept then
            count ← count + 1
if count ≥ 100 then
    return “yes”
else
    return “no”
```

Suppose, however, that there are no strings in L whose lengths are in the range n to $2n-1$. We claim there are no strings in L of lengths $2n$ or more, and thus testing all strings of lengths between 0 and $n-1$ is sufficient for us to tell whether L contains at least 100 strings. In the proof, suppose w is the shortest string in L of length at least $2n$. Then the pumping lemma applies to w, and we can write $w = xyz$, where xz is also in L. How long could xz be? It can't be as long as $2n$, because it is shorter than w, and w is the shortest string in L of length $2n$ or more. It can't be shorter than n, because $|y| \leq n$. Thus, xz is of length between n and $2n-1$, which is a contradiction, since we assumed there were no strings in L with a length in that range.

Clearly, the blackbox D and constant n can be easily determined if the input regular language is represented as a DFA (or NFA or regular expression), That is, D is basically the membership algorithm and n could be fixed as the size of the DFA.

Problem 5. (Exercise 4.4.2, 20 points)

The following figure is the transition table of a DFA.

	0	1
$\rightarrow A$	B	E
B	C	F
$*C$	D	H
D	E	H
E	F	I
$*F$	G	B
G	H	B
H	I	C
$*I$	A	E

(a) Draw the table of distinguishabilities for this automaton.

(b) Construct the minimum-state equivalent DFA.

(a)

B	$\times$							
C	$\times$	$\times$						
D		$\times$	$\times$					
E	$\times$		$\times$	$\times$				
F	$\times$	$\times$		$\times$	$\times$			
G		$\times$	$\times$		$\times$	$\times$		
H	$\times$		$\times$	$\times$		$\times$	$\times$	
I	$\times$	$\times$		$\times$	$\times$		$\times$	$\times$
	A	B	C	D	E	F	G	H

(b) Equivalent classes: $\{A, D, G\}$, $\{B, E, H\}$, $\{C, F, I\}$.