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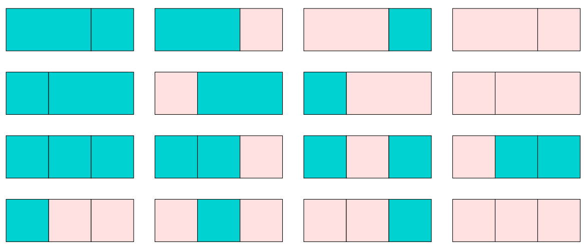
Problem 1: We have two shapes of dominoes, 1×1 squares and 2×1 rectangles. Each domino can be of one of two colors (in the figure below, dark grey or light grey.) Determine the number of ways to fully cover a $n \times 1$ rectangle with such dominoes. Dominoes cannot overlap and have to be contained in the rectangle.

For example, for $n = 3$, we get the following coverings:



A complete solution must consist of the following steps:

(a) Set up a recurrence equation.
(b) Give its characteristic polynomial and compute the roots.
(c) Give the general form of the solution.
(d) Determine the final solution.

The recurrence is

$$\begin{aligned} a_n &= 2a_{n-1} + 2a_{n-2} \\ a_0 &= 1 \\ a_1 &= 2 \end{aligned}$$

The solution is:

$$a_n = \frac{3+\sqrt{3}}{6}(1+\sqrt{3})^n + \frac{3-\sqrt{3}}{6}(1-\sqrt{3})^n$$

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Problem 2: Find x that satisfies the following congruences (use the Chinese Remainder Theorem.) Show your work.

$$\begin{aligned} x &\equiv 1 \pmod{11} \\ x &\equiv 3 \pmod{4} \\ x &\equiv 7 \pmod{13} \end{aligned}$$

We have $M = 11 \cdot 4 \cdot 13 = 572$. Compute the M_i and y_i:

	a_i	m_i	M_i	y_i
1	1	11	52	7
2	3	4	143	3
3	7	13	44	8

Then

$$\begin{aligned} x &= (1 \cdot 52 \cdot 7 + 3 \cdot 143 \cdot 3 + 7 \cdot 44 \cdot 8) \bmod 572 \\ &= (364 + 1287 + 2464) \bmod 572 \\ &= 111. \end{aligned}$$

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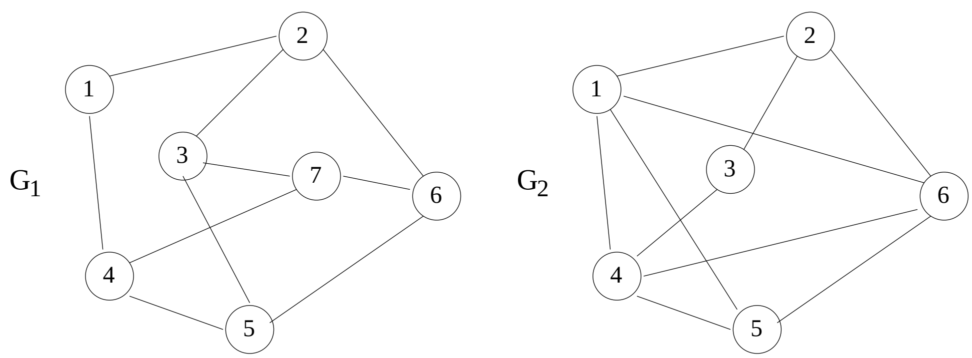
Problem 3: For the graphs below, determine the minium number of colors necessary to color them. Give an appropriate coloring (use numbers $1, 2, 3, \ldots$ for colors) and prove that there is no coloring with fewer colors. (Hint: identify subgraphs for which the number of colors is easy to determine.)



G_1 can be colored with 3 colors (easy to find). It also requires three colors because it contains a cycle of length 5 (odd.)

G_2 can be easily colored with 4 colors. It also requires 4 colors, for it contains K_4 (vertices $1, 4, 5, 6$.)

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Problem 4: Use the Θ-notation to determine the rate of growth of the following functions:

Function	big-Θ estimate
$5n^2 + \log^5 n$	$\Theta(n^2)$
$n^{100} + 2^n$	$\Theta(2^n)$
$2^{2n} + 3^n$	$\Theta(2^{2n})$
$n \log n + n^2/(\log n)$	$\Theta(n^2/\log n)$
$\sqrt{n} + 3\log^5 n$	$\Theta(\sqrt{n})$

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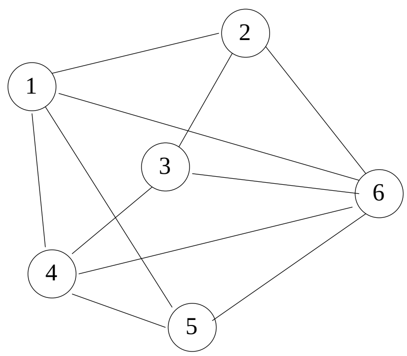
Problem 5: Recall that $\phi(n)$ denotes the Euler function, that is the number of positive integers smaller than n that are relatively prime to n. Determine the value of $\phi(1445)$. Hint: Use the factorization of 1445 and the inclusion-exclusion principle. Show your work.

Factoring, we get $1445 = 5{\cdot}289 = 5{\cdot}17^2$. Among the numbers $1, 2, ..., 1445$, there are $1445/5 = 289$ multiples of 5, $1445/17 = 85$ multiples of 17, and $1445/85 = 17$ that are multiples of both. So

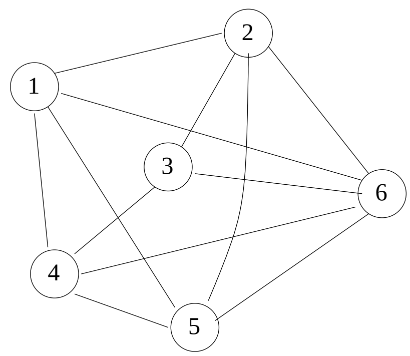
$$\phi(n) \quad = \quad 1445 - 289 - 85 + 17 \;=\; 1088.$$

Problem 6: (a) State Kuratowski's theorem.
(b) For each graph below, determine whether it is planar or not. If a graph is planar, show a planar embedding. If a graph is not planar, prove it. (You can use Euler's inequality, Kuratowski's theorem, or a direct argument.)

G_1



G_2



G_1 is planar, you can pull the edges $(1,6)$ $(1,5)$ outside. G_2 is not planar, as it contains a subgraph homeomorphic to $K_{3,3}$: one partition is $\{1,3,5\}$ and the other $\{2,4,6\}$. (Remove edges $(2,6)$, $(1,5)$, $(4,6)$, to see it better.)

Problem 7: Prove (by induction) that a binary tree of height h has at most 2^h leaves.

The proof is by induction. In the base case, for $h = 0$, we have $1 = 2^0$ leaf, so the theorem holds.

Suppose the theorem holds up to height h. We show that it also holds for height $h+1$. Take any tree T of height $h+1$, and remove all leaves. Call the new tree T'. Since T' is of height h, by the induction assumption it has at most 2^h leaves. But each leaf of T' has at most two children in T (which are leaves in T), so the number of leaves in T is at most $2 \cdot 2^h = 2^{h+1}$.

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Problem 8: Let X be the set of pairs (x_1, x_2) of integers, where $x_1 \in \{0,1\}$ and $x_2 \in \{0,1,2\}$. Define relation R on X, where $(x_1, x_2)R(y_1, y_2)$ iff $x_1^2 + x_2^2 \equiv y_1^2 + y_2^2 \pmod 3$. Give the matrix of R, determine if R is an equivalence relation, and if so, give its equivalence classes.

The matrix of R:

	$(0,0)$	$(0,1)$	$(0,2)$	$(1,0)$	$(1,1)$	$(1,2)$
$(0,0)$	1	0	0	0	0	0
$(0,1)$	0	1	1	1	0	0
$(0,2)$	0	1	1	1	0	0
$(1,0)$	0	1	1	1	0	0
$(1,1)$	0	0	0	0	1	1
$(1,2)$	0	0	0	0	1	1

From the matrix, R is reflexive, symmetric and transitive, so it is an equivalence relation. Its equivelence classes are $\{(0,0)\}$, $\{(0,1),(0,2),(1,0)\}$, $\{(1,1),(1,2)\}$.