NAME: SID:

Problem 1: Use the Θ-notation to determine the rate of growth of the following functions:

Function	Θ estimate
$5n + 3n^2 + 3$	$\Theta(n^2)$
$17n + 3n^2 \log n + 1$	$\Theta(n^2 \log n)$
$7n^9 + (1.5)^n$	$\Theta((1.5)^n)$
$n^3 4^n + 5^n + 16\sqrt{n}$	$\Theta(5^n)$
$\sqrt{n} + 11 \log n$	$\Theta(\sqrt{n})$

Problem 2: (a) State Euclid's Algorithm.

(b) Use Euclid's Algorithm to compute the greatest common divisor of 391 and 299. Show your work.

Solution: We know that $gcd(a, b) = gcd(b, a - b) = gcd(b, a \operatorname{rem} b)$. Note that, using $a \operatorname{rem} b$ is more efficient than using $a - b$.

$$\begin{aligned} gcd(391, 299) &= gcd(299, 391 \operatorname{rem} 299) \\ &= gcd(299, 92) \\ &= gcd(92, 299 \operatorname{rem} 92) \\ &= gcd(92, 23) \\ &= gcd(23, 92 \operatorname{rem} 23) \\ &= gcd(23, 0) \\ &= 23 \end{aligned}$$

Problem 3: (a) Give the factorization of 1386. Show your work.

Solution:

$$\begin{aligned}1386 &= 2 \cdot 693 \\ &= 2 \cdot 3 \cdot 231 \\ &= 2 \cdot 3 \cdot 3 \cdot 77 \\ &= 2 \cdot 3^2 \cdot 7 \cdot 11\end{aligned}$$

(b) Determine $10^{-1} \pmod{13}$, the inverse of 10 modulo 13. Show your work.

Solution: We want to find integers x and y that satisfy: $10 \cdot x + 13 \cdot y = 1$. Since, 10 is relatively prime to 13, such integers should exist.

$$\begin{aligned}10 \cdot x &= 10, 20, 30, \mathbf{40}, 50, 60, \ldots \\ 13 \cdot y &= 13, 26, \mathbf{39}, 52, 65, 78, \ldots\end{aligned}$$

So, for $x = 4$ and $y = -3$, $10 \cdot x + 13 \cdot y = 40 - 39 = 1$. So, $10^{-1} \pmod{13} = 4$.

Alternatively, since 13 is a prime number, $10^{13-1} \equiv 1 \pmod{13}$ (Fermat's Little Theorem). So, $10^{11} \cdot 10 \equiv 1 \pmod{13}$, meaning the inverse is $10^{11} \pmod{13}$.

$$\begin{array}{lllll}10 & \equiv 10 & & & \pmod{13} \\ 10^2 & \equiv 10^2 & \equiv 9 & & \pmod{13} \\ 10^4 & \equiv 9^2 & \equiv 3 & & \pmod{13} \\ 10^8 & \equiv 3^2 & \equiv 9 & & \pmod{13} \\ 10^{10} & \equiv 10^2 \cdot 10^8 & \equiv 9 \cdot 9 & \equiv 3 & \pmod{13} \\ 10^{11} & \equiv 10 \cdot 10^{10} & \equiv 10 \cdot 3 & \equiv 4 & \pmod{13}\end{array}$$