Greedy Algorithms

Chapter 16
Optimization Problems

- A class of problems in which we are asked to find a set (or a sequence) of “items” that satisfy some constraints and simultaneously optimize (i.e., maximize or minimize) some objective function
Example: Bin Packing
Example: Bin Packing
Example: Bin Packing
Example: Bin Packing

5

6

1

3

4

2
Example: Bin Packing
Example: Bin Packing

Number of bins = 3
Example: Bin Packing

(Optimal Solution) Number of bins = 2
The Greedy Method

- Applied to optimization problems
- Adds items to the solution one-by-one
- Builds up towards the final solution
- No backtracking

- Not necessarily optimal!
Activity Selection

a.k.a. Task Scheduling
Activity Selection Problem

Given a set of activities $S = \{a_1, a_2, \ldots, a_n\}$ where each activity $i$ has a start time $s_i$ and a finish time $f_i$, where $0 \leq s_i < f_i < \infty$. An activity $a_i$ happens in the half-open time interval $[s_i, f_i)$. Two activities are said to be **compatible** if they do not overlap. The problem is to find a **maximum-size compatible subset**, i.e., a one with the maximum number of activities.
Example of Activity Selection
A Solution

- a3[0,6)
- a10[2,14)
- a1[1,4)
- a9[8,12)
- a5[3,9)
- a4[5,7)
- a8[8,11)
- a2[3,5)
- a7[6,10)
- a11[12,16)
- a6[5,9)
A Better Solution

- \( a_3[0,6] \)
- \( a_{10}[2,14] \)
- \( a_1[1,4] \)
- \( a_9[8,12] \)
- \( a_5[3,9] \)
- \( a_4[5,7] \)
- \( a_8[8,11] \)
- \( a_2[3,5] \)
- \( a_7[6,10] \)
- \( a_{11}[12,16] \)
- \( a_6[5,9] \)
An Optimal (Best) Solution
Another Optimal Solution
“Greedy” Strategies

› Longest first
› Shortest first
› Early start first
› Early finish first
› …
Early Finish Greedy Strategy

1. Sort activities by finish time
2. Schedule the first activity
3. Remove all incompatible activities
4. If there are more activities, repeat 2
Early Finish

a3[0,6)

a10[2,14]

a1[1,4] a9[8,12]

a5[3,9] a8[8,11]

a4[5,7] a7[6,10]

a2[3,5] a11[12,16]

a6[5,9]

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Early Finish

- a3[0, 6]
- a10[2, 14]
- a1[1, 4]
- a9[8, 12]
- a5[3, 9]
- a4[5, 7]
- a8[8, 11]
- a2[3, 5]
- a7[6, 10]
- a11[12, 16]
- a6[5, 9]
Early Finish

- $a_1[1,4)$
- $a_9[8,12)$
- $a_4[5,7)$
- $a_8[8,11)$
- $a_7[6,10)$
- $a_{11}[12,16)$
- $a_6[5,9)$
Early Finish

\[
\begin{align*}
a_1[1,4] & \quad a_9[8,12] \\
a_4[5,7] & \quad a_8[8,11] \\
a_7[6,10] & \quad a_{11}[12,16] \\
a_6[5,9] & 
\end{align*}
\]
Early Finish

Early Finish

a1[1,4]       a9[8,12]

a4[5,7]       a8[8,11]

a11[12,16]

\( t \)
Early Finish

a1[1,4]
a4[5,7]  a8[8,11]
a11[12,16]
Early Finish

\[ a_1[1,4] \]

\[ a_4[5,7] \]

\[ a_8[8,11] \]

\[ a_{11}[12,16] \]
Optimality of the Greedy Choice

To prove optimality of the greedy choice, we have to prove the following two properties

1. **Greedy Choice**: The greedy choice is part of the answer

2. **Optimal Substructure**: The optimal solution to the big problem contains the optimal solution to the sub-problem
Greedy Choice

- Let $A \subseteq S$ be an optimal solution. Let $a_j$ be the first element in $A$ and $a_m$ be the first element in $S$.
- We want to prove that $a_m$ is part of an optimal solution.
- If $a_m = a_j$ then we are done
- Otherwise, we prove that there is another optimal solution $A^\prime = A \setminus \{a_j\} \cup \{a_m\}$
- Is $A^\prime$ a solution? yes
- Is $A^\prime$ optimal? yes
Optimal Substructure

- We want to prove that, if $A \subseteq S$ is an optimal solution to $S$, then $A - \{a_m\}$ is an optimal solution to $S' = S - \{a_j : s_j > f_1\}$

Proof by contradiction

- Assume that $A - \{a_m\}$ is not optimal

- Then, there is another solution $B$ to $S'$ such that $|B| > |A - \{a_m\}| \geq |A|$

- If this is the case, then the subset $A' = \{a_m\} \cup B$ is also a solution to $S$

- $|A'| > |B| \Rightarrow |A'| > |A|$, which means that $A$ is not optimal which is a contradiction