$O$-notation

$\exists c > 0, n_0 > 0$

$0 \leq f(n) \leq cg(n)$

$n \geq n_0$

g(n) is an asymptotic upper bound for f(n)
\[ \exists c > 0, n_0 > 0 \]
\[ 0 \leq cg(n) \leq f(n) \]
\[ n \geq n_0 \]

\[ g(n) \text{ is an asymptotic lower-bound for } f(n) \]
$\Theta$-notation

\[ \exists c_1, c_2 > 0, n_0 > 0 \\
0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n) \\
n \geq n_0 \]

$g(n)$ is an asymptotic tight bound for $f(n)$
The image contains a mathematical expression and a graph. The expression is:

\[ f(n) = o(g(n)) \]

This expression indicates that \( f(n) \) is a non-tight asymptotic upper bound for \( g(n) \). The graph shows several curves, each labeled as \( c_1 g(n) \), \( c_2 g(n) \), \( c_3 g(n) \), etc., indicating different constants \( c_i \) multiplying \( g(n) \). The graph also shows a curve labeled \( f(n) \) that is asymptotically upper bounded by \( g(n) \).
\omega\text{-notation}

\forall c > 0 \\
\exists n_0 > 0 \\
0 \leq cgn(n) \leq f(n) \\
n \geq n_0

g(n) \text{ is a non-tight asymptotic lower-bound for } f(n)
Compare two functions

\[ \lim_{n \to \infty} \frac{f(n)}{g(n)} \]

- 0: \( f(n) = o(g(n)) \)
- \( c > 0 \): \( f(n) = \Theta(g(n)) \)
- \( \infty \): \( f(n) = \omega(g(n)) \)
## Analogy to real numbers

<table>
<thead>
<tr>
<th>Functions</th>
<th>Real numbers</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(n) = O(g(n))$</td>
<td>$a \leq b$</td>
</tr>
<tr>
<td>$f(n) = \Omega(g(n))$</td>
<td>$a \geq b$</td>
</tr>
<tr>
<td>$f(n) = \Theta(g(n))$</td>
<td>$a = b$</td>
</tr>
<tr>
<td>$f(n) = o(g(n))$</td>
<td>$a &lt; b$</td>
</tr>
<tr>
<td>$f(n) = \omega(g(n))$</td>
<td>$a &gt; b$</td>
</tr>
</tbody>
</table>
Simple Rules

- We can omit constants
- We can omit lower order terms
- $\Theta(an^2 + bn + c)$ becomes $\Theta(n^2)$
- $\Theta(c_1)$ and $\Theta(c_2)$ become $\Theta(1)$
- $\Theta(\log_{k_1} n)$ and $\Theta(\log_{k_2} n)$ become $\Theta(\lg n)$
- $\Theta(\lg(n^k))$ becomes $\Theta(\lg n)$
- $\lg^{k_1}(n) = o\left(n^{k_2}\right)$ for any positive constants $k_1$ and $k_2$
Popular Classes of Functions

- **Constant:** \( f(n) = \Theta(1) \)
- **Logarithmic:** \( f(n) = \Theta(\lg(n)) \)
- **Sublinear:** \( f(n) = o(n) \)
- **Linear:** \( f(n) = \Theta(n) \)
- **Super-linear:** \( f(n) = \omega(n) \)
- **Quadratic:** \( f(n) = \Theta(n^2) \)
- **Polynomial:** \( f(n) = \Theta(n^k); \ k \text{ is a constant} \)
- **Exponential:** \( f(n) = \Theta(k^n); \ k \text{ is a constant} \)
**Insertion Sort (Revisit)**

**Insertion-Sort** $(A, n)$

```plaintext
for $j = 2$ to $n$
    key = $A[j]$
    $i = j - 1$
    while $i > 0$ and $A[i] > key$
        $A[i + 1] = A[i]$
        $i = i - 1$
    $A[i + 1] = key$
```

$\Theta(n^2)$

n-times

j-times