Problem 1: Compute $S_4$ given by Simpson’s Rule for $\int_1^4 \ln x \, dx$. You do not need to simplify or approximate your result.

Note that $\Delta x = \frac{4 - 1}{4} = \frac{3}{4}$.

$$S_4 = \frac{1}{3} \Delta x (ln 1 + 4 \ln(1 + \Delta x) + 2 \ln(1 + 2\Delta x) + 4 \ln(1 + 3\Delta x) + \ln(1 + 4\Delta x))$$

$$= \frac{1}{4} \left( \ln 1 + 4 \ln \frac{7}{4} + 2 \ln \frac{5}{2} + 4 \ln \frac{13}{4} + \ln 4 \right)$$

For reference, this leads to $S_4 \approx 2.543$ compared to the exact answer of $8 \ln 2 - 3 \approx 2.545$.

Problem 2: Show that $\int_{-\infty}^{\infty} \frac{dx}{(x^2 + 1)^{3/2}}$ converges and evaluate it.

The integral converges since

$$\int_{-R}^{R} \frac{dx}{(x^2 + 1)^{3/2}} \leq \int_{-1}^{1} \frac{dx}{(x^2)^{3/2}} + \int_{0}^{R} \frac{dx}{(x^2 + 1)^{3/2}} = 2 + \int_{-R}^{R} \frac{dx}{x^3} + \int_{0}^{R} \frac{dx}{x^3}$$

converges. To evaluate, use $x = \tan u, dx = \sec^2 u \, du$

$$\int \frac{dx}{(x^2 + 1)^{3/2}} = \int \frac{\sec^2 u \, du}{(\tan^2 u + 1)^{3/2}} = \int \frac{\sec^2 u \, du}{\sec^3 u} = \int \cos u \, du = \sin u + C = \frac{x}{\sqrt{x^2 + 1}} + C$$

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2 + 1)^{3/2}} = \lim_{R \to \infty} \int_{R}^{\infty} \frac{dx}{(x^2 + 1)^{3/2}} = \lim_{R \to \infty} \left[ \frac{x}{\sqrt{x^2 + 1}} \right]_{-R}^{R}$$

$$= \lim_{R \to \infty} \left( \frac{R}{\sqrt{R^2 + 1}} - \frac{-R}{\sqrt{(-R)^2 + 1}} \right) = \lim_{R \to \infty} \left( \frac{1}{\sqrt{1 + R^{-2}}} - \frac{-1}{\sqrt{1 + R^{-2}}} \right) = 2$$

The convergence argument could also be made on the grounds that the limits converge. Note that it is valid to compute one limit for both the upper and lower limits of integration since I have already established that the integral will converge. If the convergence of this limit is to be used for this purpose, then two limits must be computed separately and both must converge.
Problem 3: Show that \( \int_{0}^{3} \frac{dx}{\sqrt{9-x^2}} \) converges and evaluate it.

Evaluate the definite integral
\[
\int \frac{dx}{\sqrt{9-x^2}} = \int \frac{3 \cos u \, du}{\sqrt{9 - (3 \sin u)^2}}
\]
\[= \int \frac{\cos u \, du}{\cos u} = \int du \]
\[= u + C = \sin^{-1} \frac{x}{3} + C\]

Convergence follows from the existence of the limit
\[
\lim_{R \to 3} \int_{0}^{R} \frac{dx}{\sqrt{9-x^2}} = \lim_{R \to 3} \left[ \sin^{-1} \frac{x}{3} \right]_{0}^{R} = \sin^{-1} 3 - \sin^{-1} 0 = \frac{\pi}{2}
\]

Problem 4: Calculate a bound for the error on \( M_{10} \) for \( \int_{1}^{4} \ln x \, dx \). Do not calculate \( M_{10} \).

The error bound for midpoint rule is \( E \leq \frac{K_2(b-a)^3}{24N^2} \), where here \( a = 1, b = 4, \) and \( N = 10 \). To get \( K_2 \), note that \( f(x) = \ln x, \ f'(x) = x^{-1}, \) and \( f''(x) = -x^{-2} \). Since \( |f''(x)| = x^{-2} \) is decreasing on \([1,4]\), the maximum is obtained at the left endpoint, and \( K_2 = 1 \). The error bound is
\[
E \leq \frac{K_2(b-a)^3}{24N^2} = \frac{9}{800}.
\]

This bound is approximately 0.0113, compared to the actual error in the approximation 0.00279.