Problem 1: Evaluate \( \lim_{{t \to 0^+}} (\sin t)(\ln t) \).

Note that \( \ln t \to -\infty \) and \( \sin t \to 0 \), which suggests we may be able to get this into a form where L'Hôpital's rule can be applied. One way to do this is

\[
\lim_{{t \to 0^+}} (\sin t)(\ln t) = \lim_{{t \to 0^+}} \frac{\ln t}{\csc t} = \lim_{{t \to 0^+}} \frac{t^{-1}}{-\cot t \csc t} = -\lim_{{t \to 0^+}} \frac{\sin t \tan t}{t}.
\]

At this point L'Hôpital's rule can be applied directly, but breaking the limit into a product of limits makes this step simpler

\[
\lim_{{t \to 0^+}} (\sin t)(\ln t) = -\lim_{{t \to 0^+}} \frac{\sin t \tan t}{t} = -\left( \lim_{{t \to 0^+}} \frac{\sin t}{t} \right) \left( \lim_{{t \to 0^+}} \tan t \right) = -(1)(0) = 0.
\]

Problem 2: Evaluate \( \int e^{-5a} \sin a \, da \).

The key to this problem is to use integration by parts twice.

\[
\int e^{-5a} \sin a \, da = e^{-5a} (\cos a) - \int (-5e^{-5a}) (\cos a) \, da = e^{-5a} \cos a - \frac{5}{26} \int e^{-5a} \cos a \, da
\]

\[
= -e^{-5a} \cos a - \frac{5}{26} \left( e^{-5a} \sin a - \int (-5e^{-5a}) \sin a \, da \right)
\]

\[
= -e^{-5a} \cos a - \frac{5}{26} e^{-5a} \sin a - 25 \int e^{-5a} \sin a \, da
\]

\[
26 \int e^{-5a} \sin a \, da = -e^{-5a} \cos a - 5e^{-5a} \sin a + C
\]

\[
\int e^{-5a} \sin a \, da = -\frac{1}{26} e^{-5a} \cos a - \frac{5}{26} e^{-5a} \sin a + C_2
\]
Problem 3: Evaluate: \( \int_0^3 xe^{4x} \, dx \).

Integration by parts is the key here.

\[
\int xe^{4x} \, dx = x \left( \frac{1}{4} e^{4x} \right) - \int \left( \frac{1}{4} e^{4x} \right) \, dx = \frac{1}{4} xe^{4x} - \frac{1}{4} \int e^{4x} \, dx = \frac{1}{4} xe^{4x} - \frac{1}{16} e^{4x} + C
\]

Then, the definite integral can be computed

\[
\int_0^3 xe^{4x} \, dx = \left[ \frac{1}{4} xe^{4x} - \frac{1}{16} e^{4x} \right]_0^3 = \frac{1}{4} (3) e^{4(3)} - \frac{1}{16} e^{4(3)} - \frac{1}{4} (0) e^{4(0)} + \frac{1}{16} e^{4(0)} = \frac{11}{16} e^{12} + \frac{1}{16}
\]

The two steps may also be done at the same time.

Problem 4: Find the minimum initial deposit \( P_0 \) necessary to fund an annuity for \( T \) years if withdrawals are made at a rate of \( w \) and interest is earned at a rate of \( r \). Assume that withdrawals are made continuously, and interest is compounded continuously.

Let \( P(t) \) be the value of the annuity at time \( t \) (years). With interest rate \( r \) alone, we would have \( P' = rP \). Adding in withdrawal rate \( w \), we have \( P' = rP - w \). This gives us a solution of the form \( P = ae^{rt} + b \). Differentiating and plugging back into the differential equation,

\[
P' = rae^{rt} = rP - w = r( ae^{rt} + b ) - w = rae^{rt} + rb - w.
\]

From this we have \( b = \frac{w}{r} \). Thus, \( P = ae^{rt} + \frac{w}{r} \). From \( P(0) = a + \frac{w}{r} = P_0 \), \( a = P_0 - \frac{w}{r} \).

\( P = (P_0 - \frac{w}{r}) e^{rt} + \frac{w}{r} \). At worst, the money will run out after \( T \) years, so

\[
\left( P_0 - \frac{w}{r} \right) e^{rT} + \frac{w}{r} = 0
\]

\[
\left( P_0 - \frac{w}{r} \right) e^{rT} = -\frac{w}{r}
\]

\[
P_0 - \frac{w}{r} = -\frac{w}{r} e^{-rT}
\]

\[
P_0 = \frac{w}{r} (1 - e^{-rT})
\]