$$
\begin{aligned}
(x+1)y' + y &= x \\
y &= \sum_{n=0}^{\infty} a_n x^n \\
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
&= \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n \\
xy' &= \sum_{n=1}^{\infty} n a_n x^n \\
xy' + y' + y &= x \\
\sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n &= x \\
\sum_{n=1}^{\infty} n a_n x^n + \sum_{n=1}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=1}^{\infty} a_n x^n &= x - a_1 - a_0 \\
\sum_{n=1}^{\infty} (n a_n + a_n + (n+1) a_{n+1}) x^n &= x - a_1 - a_0 \\
\sum_{n=1}^{\infty} (n+1)(a_n + a_{n+1}) x^n &= x - a_1 - a_0 \\
\sum_{n=2}^{\infty} (n+1)(a_n + a_{n+1}) x^n &= x - 2(a_1 + a_2)x - a_1 - a_0
\end{aligned}
$$

$$\sum_{n=2}^{\infty}(n+1)(a_n+a_{n+1})x^n = x-2(a_1+a_2)x-a_1-a_0$$

$$\begin{aligned}
a_1 &= -a_0 \\
2(a_1+a_2) &= 1 \\
a_1+a_2 &= \frac{1}{2} \\
a_2 &= \frac{1}{2}-a_1 \\
a_2 &= \frac{1}{2}+a_0 \\
a_{n+1} &= -a_n \quad n \geq 2 \\
a_n &= (-1)^{n-2}a_2 \\
a_n &= (-1)^n\left(\frac{1}{2}+a_0\right) \\
y &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n \\
&= a_0 - a_0 x + \sum_{n=2}^{\infty}(-1)^n\left(\frac{1}{2}+a_0\right)x^n \\
&= a_0\left(1-x+\sum_{n=2}^{\infty}(-1)^n x^n\right) + \sum_{n=2}^{\infty}(-1)^n\frac{1}{2}x^n \\
&= a_0\sum_{n=0}^{\infty}(-1)^n x^n + \frac{1}{2}\sum_{n=0}^{\infty}(-1)^{n+2}x^{n+2}
\end{aligned}$$

$$
\begin{aligned}
y &= a_0 \sum_{n=0}^{\infty} (-1)^n x^n + \frac{1}{2} \sum_{n=0}^{\infty} (-1)^{n+2} x^{n+2} \\
&= a_0 \sum_{n=0}^{\infty} (-1)^n x^n + \frac{x^2}{2} \sum_{n=0}^{\infty} (-1)^n x^n \\
&= \left(a_0 + \frac{x^2}{2} \right) \sum_{n=0}^{\infty} (-1)^n x^n \\
&= \frac{a_0 + \frac{x^2}{2}}{1+x} \\
\frac{1}{1-z} &= \sum_{n=0}^{\infty} z^n
\end{aligned}
$$

$$
\begin{aligned}
xy' + y &= 0 \\
y &= \sum_{n=0}^{\infty} a_n x^n \\
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
xy' &= \sum_{n=1}^{\infty} n a_n x^n \\
\sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n &= 0 \\
\sum_{n=1}^{\infty} (n+1) a_n x^n &= 0 \\
y &= \frac{a}{x}
\end{aligned}
$$

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= 1 \\
F_n &= F_{n-1} + F_{n-2} \\
R(x) &= \sum_{n=0}^{\infty} F_n x^n \\
xR(x) &= \sum_{n=0}^{\infty} F_n x^{n+1} \\
xR(x) &= \sum_{n=1}^{\infty} F_{n-1} x^n \\
x^2 R(x) &= \sum_{n=1}^{\infty} F_{n-1} x^{n+1} \\
x^2 R(x) &= \sum_{n=2}^{\infty} F_{n-2} x^n \\
R(x) - xR(x) - x^2 R(x) &= \sum_{n=0}^{\infty} F_n x^n - \sum_{n=1}^{\infty} F_{n-1} x^n - \sum_{n=2}^{\infty} F_{n-2} x^n \\
R(x) - xR(x) - x^2 R(x) &= F_0 + F_1 x + \sum_{n=2}^{\infty} F_n x^n - F_0 x - \sum_{n=2}^{\infty} F_{n-1} x^n - \sum_{n=2}^{\infty} F_{n-2} x^n \\
R(x) - xR(x) - x^2 R(x) &= x + \sum_{n=2}^{\infty} (F_n - F_{n-1} - F_{n-2}) x^n \\
(1 - x - x^2) R(x) &= x \\
R(x) &= \frac{x}{1 - x - x^2} \\
\frac{x}{1 - x - x^2} &= \sum_{n=0}^{\infty} F_n x^n
\end{aligned}
$$

$$
\begin{aligned}
r_{\pm} &= \frac{1 \pm \sqrt{5}}{2} \\
\frac{x}{1-x-x^2} &= \frac{-x}{(x+r_-)(x+r_+)} \\
\frac{x}{1-x-x^2} &= \frac{A}{x+r_-} + \frac{B}{x+r_+} \\
\frac{x}{1-x-x^2} &= \frac{\frac{A}{r_-}}{1+\frac{x}{r_-}} + \frac{\frac{B}{r_+}}{1+\frac{x}{r_+}} \\
\frac{x}{1-x-x^2} &= \frac{A}{r_-}\sum_{n=0}^{\infty}\left(-\frac{x}{r_-}\right)^n + \frac{B}{r_+}\sum_{n=0}^{\infty}\left(-\frac{x}{r_+}\right)^n \\
\frac{x}{1-x-x^2} &= \sum_{n=0}^{\infty}\left(\frac{A}{r_-}\left(-\frac{x}{r_-}\right)^n + \frac{B}{r_+}\left(-\frac{x}{r_+}\right)^n\right) \\
\frac{x}{1-x-x^2} &= \sum_{n=0}^{\infty}\left(\frac{A}{r_-}(xr_+)^n + \frac{B}{r_+}(xr_-)^n\right) \\
\frac{x}{1-x-x^2} &= \sum_{n=0}^{\infty}\left(\frac{A}{r_-}r_+^n + \frac{B}{r_+}r_-^n\right)x^n \\
F_n &= \frac{A}{r_-}r_+^n + \frac{B}{r_+}r_-^n \\
F_n &= \frac{r_-^n + r_+^n}{\sqrt{5}} \\
r_-r_+ &= -1
\end{aligned}
$$