Midterm 1

Name: ___________________________  ID: ___________  Section: ___

You have 50 minutes to complete this quiz. You must show your work to receive credit. There are more problems on the back.

Problem 1 (10 points): Evaluate: \( \int e^{-x} \sin x \, dx \) and \( \int e^{-x} \cos x \, dx \).

Use integration by parts twice to get one of them.

\[
\int e^{-x} \sin x \, dx = e^{-x}(- \cos x) - \int (-e^{-x})(- \cos x) \, dx \\
= -e^{-x} \cos x - \int e^{-x} \cos x \, dx \\
= -e^{-x} \cos x - e^{-x} \sin x + \int (-e^{-x}) \sin x \, dx
\]

\[
\int e^{-x} \sin x \, dx = -\frac{1}{2}e^{-x} \cos x + \frac{1}{2}e^{-x} \sin x + C
\]

Then substitute back to get the other.

\[
\int e^{-x} \cos x \, dx = -e^{-x} \cos x - \int e^{-x} \sin x \, dx \\
= -e^{-x} \cos x - \left(-\frac{1}{2}e^{-x} \cos x - \frac{1}{2}e^{-x} \sin x + C\right) \\
= -\frac{1}{2}e^{-x} \cos x + \frac{1}{2}e^{-x} \sin x + C_2
\]

Problem 2 (10 points): Evaluate: \( \int x e^{-x} \sin x \, dx \).

Use integration by parts again, reusing the results of previous problems.

\[
u = x \quad du = dx \quad dv = e^{-x} \sin x \, dx \quad v = -\frac{1}{2}e^{-x} \cos x - \frac{1}{2}e^{-x} \sin x
\]
\[
\int xe^{-x} \sin x \, dx = x\left(-\frac{1}{2}e^{-x}\cos x - \frac{1}{2}e^{-x}\sin x\right) - \int \left(-\frac{1}{2}e^{-x}\cos x - \frac{1}{2}e^{-x}\sin x\right) \, dx \\
= -\frac{1}{2}xe^{-x}\cos x - \frac{1}{2}xe^{-x}\sin x + \frac{1}{2} \int e^{-x}\cos x \, dx + \frac{1}{2} \int e^{-x}\sin x \, dx \\
\int xe^{-x} \sin x \, dx = -\frac{1}{2}xe^{-x}\cos x - \frac{1}{2}xe^{-x}\sin x - \frac{1}{2}e^{-x}\cos x + C
\]

**Problem 3 (10 points): Evaluate:** \[\lim_{x \to 0^+} \frac{x^{2x} - 1}{x \ln x}\]

The denominator tends to zero. To see if the numerator does, I need to know \(\lim_{x \to 0^+} x^{2x}\).

\[
\ln \left(\lim_{x \to 0^+} x^{2x}\right) = \lim_{x \to 0^+} \ln(x^{2x}) = \lim_{x \to 0^+} 2x \ln x = 2 \lim_{x \to 0^+} \frac{\ln x}{x^{-1}} = 2 \lim_{x \to 0^+} \frac{x^{-1}}{-x^{-2}} = -2 \lim_{x \to 0^+} x = 0
\]

Thus, \(\lim_{x \to 0^+} x^{2x} = e^0 = 1\), and the numerator is also zero. Now I can apply L'Hôpital’s rule, which means I must now differentiate \(x^{2x}\), which I can do with logarithmic differentiation.

\[
y = x^{2x} \\
\ln y = \ln(x^{2x}) = 2x \ln x \\
\frac{y'}{y} = 2 \ln x + 2 \frac{x}{x} = 2(\ln x + 1) \\
y' = 2(\ln x + 1)x^{2x}
\]

From here, L'Hôpital’s rule finishes things up pretty quickly.

\[
\lim_{x \to 0^+} \frac{x^{2x} - 1}{x \ln x} = \lim_{x \to 0^+} \frac{2(\ln x + 1)x^{2x}}{\ln x + 1} = 2 \lim_{x \to 0^+} x^{2x} = 2
\]

**Problem 4 (10 points): For a falling object of mass \(m\), free-fall with air resistance can be modeled with \(v' = -\frac{k}{m}(v + \frac{mg}{k})\). The object starts at rest and tends towards a terminal velocity \(v_1\). How long does it take the object to reach half its terminal velocity? (Note that \(v_1 < 0\), since the object is falling.)**

At terminal velocity, \(v' = 0\), so that \(v_1 = -\frac{mg}{k}\), or \(k = -\frac{mg}{v_1}\). The object’s velocity will be \(v = Ce^{-\frac{k}{m}t} - \frac{mg}{k}\). Since \(v(0) = 0\), \(C = \frac{mg}{k}\). Finally,

\[
v = \left(e^{-\frac{k}{m}t} - 1\right)\frac{mg}{k} \\
= -\left(e^{\frac{mg}{v_1}t} - 1\right)v_1
\]
The time $T$ required to reach half terminal velocity $\frac{v_1}{2}$ is

$$- \left( e^{\frac{g}{v_1}} T - 1 \right) v_1 = \frac{v_1}{2}$$

$$e^{\frac{g}{v_1}} T - 1 = -\frac{1}{2}$$

$$e^{\frac{g}{v_1} T} = \frac{1}{2}$$

$$\frac{g}{v_1} T = -\ln 2$$

$$T = -\frac{v_1 \ln 2}{g}$$

**Problem 5 (10 points):** Evaluate: $\int_0^\pi \frac{\sin x}{2 \cos x + 3} \, dx$

Use the substitution $u = 2 \cos x + 3$, $du = -2 \sin x \, dx$. $x = 0 \implies u = 2 \cos 0 + 3 = 5$. $x = \pi \implies u = 2 \cos \pi + 3 = 1$.

$$\int_0^\pi \frac{\sin x}{2 \cos x + 3} \, dx = -\frac{1}{2} \int_5^1 \frac{du}{u} = -\frac{1}{2} \left[ \ln |u| \right]^1_5 = -\frac{1}{2} \ln 1 - \ln 5 = \frac{1}{2} \ln 5$$

**Problem 6 (10 points):** Given the function $f(x) = xe^x$. (2 points each)

(a) Identify the critical points.

(b) Use the second derivative test to identify each critical point as a local minimum or local maximum.

(c) Identify the inflection points.

(d) Evaluate the limits $\lim_{x \to 0} f(x)$, $\lim_{x \to \infty} f(x)$, and $\lim_{x \to -\infty} f(x)$.

(e) Use this information to sketch the function.

First, let’s compute the derivatives.

$$f'(x) = \frac{d}{dx}(xe^x) = xe^x + e^x = (x + 1)e^x$$

$$f''(x) = \frac{d}{dx}((x + 1)e^x) = (x + 1)e^x + e^x = (x + 2)e^x$$

(a) We require $f'(x) = 0$. Since $e^x > 0$, the only critical point is $x = -1$.

(b) Since $f''(-1) = ((-1) + 2)e^{-1} = e^{-1} > 0$, the critical point is a local minimum.

(c) We require $f''(x) = 0$. The only inflection point is $x = -2$. 

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(d) The first limit is $\lim_{x \to 0} xe^x = (0)e^0 = 0$, and the second limit is $\lim_{x \to \infty} xe^x = \infty$. The third limit can be evaluated with L'Hôpital’s rule.

$$\lim_{x \to -\infty} xe^x = \lim_{x \to -\infty} \frac{x}{e^{-x}} = \lim_{x \to -\infty} \frac{1}{-e^{-x}} = \lim_{x \to -\infty} -e^x = 0.$$ 

(e) The function is, with some of its features labeled,