Problem 1

Construct a traffic following model with a density-velocity relationship \( \hat{u}(\rho) \) such that drivers try to maintain a fixed following time \( T \). (For example, if \( T = 2s \), drivers always try to stay two seconds behind the car in front.)

Maintaining a following time of \( T \) implies \( T v_k(t) = x_{k-1}(t) - x_k(t) = \frac{1}{\rho} \), so that \( \hat{v}(\rho) = \frac{1}{T\rho} \).
Problem 2

Consider the linear car-following model,

\[
\frac{d^2 x_n}{dt^2}(t + T) = -\lambda \left( \frac{dx_n}{dt}(t) - \frac{dx_{n-1}}{dt}(t) \right),
\]

with a response time \( T \) (a delay). Assume the lead driver’s velocity varies periodically

\[ v_0 = \text{Re}(1 + f_0 e^{i\omega t}). \]

Also assume the \( n \)-th driver’s velocity varies periodically

\[ v_n = \text{Re}(1 + f_n e^{i\omega t}), \]

where \( f_n \) measures the amplification or decay which occurs. Then,

\[ f_n = \left( 1 + \frac{i\omega}{\lambda} e^{i\omega T} \right)^{-n} f_0. \]

Show the magnitude of the amplification factor \( f_n \) decreases with \( n \) if

\[ \frac{\sin \omega T}{\omega} < \frac{1}{2\lambda}. \]

The magnitude of amplification will decrease with \( n \) if

\[
1 > \left| 1 + \frac{i\omega}{\lambda} e^{i\omega T} \right|,
\]

\[
1 < \left| 1 + \frac{i\omega}{\lambda} e^{i\omega T} \right|^2
\]

\[
1 < \left| 1 + \frac{i\omega}{\lambda} e^{i\omega T} \right|^4
\]

\[
= \left| 1 + \frac{i\omega}{\lambda} (\cos \omega T + i \sin \omega T) \right|^2
\]

\[
= \left( 1 - \frac{\omega}{\lambda} \sin \omega T \right)^2 + \left( \frac{\omega}{\lambda} \cos \omega T \right)^2
\]

\[
= 1 - 2 \frac{\omega}{\lambda} \sin \omega T + \frac{\omega^2}{\lambda^2} \sin^2 \omega T + \frac{\omega^2}{\lambda^2} \cos^2 \omega T
\]

\[
= 1 - 2 \frac{\omega}{\lambda} \sin \omega T + \frac{\omega^2}{\lambda^2} \]

\[
\frac{2 \omega}{\lambda} \sin \omega T < \frac{\omega^2}{\lambda^2}
\]

\[
\frac{\sin \omega T}{\omega} < \frac{1}{2\lambda}
\]