Problem 1 (15 points)

A compressible ideal gas in one dimension is described by the equations

\[
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho u) = 0
\]

\[
\frac{\partial}{\partial t} (\rho u) + \frac{\partial}{\partial x} (\rho u^2 + p) = 0
\]

\[
\frac{\partial E}{\partial t} + \frac{\partial}{\partial x} ((E + p)u) = 0
\]

\[
p = \rho e (\gamma - 1)
\]

\[
E = \frac{1}{2} \rho u^2 + \rho e
\]

Fill in the table with the units of the remaining quantities.

<table>
<thead>
<tr>
<th>name</th>
<th>units</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x)</td>
<td>m</td>
</tr>
<tr>
<td>(t)</td>
<td>s</td>
</tr>
<tr>
<td>(\rho)</td>
<td>kg m(^{-1})</td>
</tr>
<tr>
<td>(u)</td>
<td>m s(^{-1})</td>
</tr>
<tr>
<td>(p)</td>
<td>kg m s(^{-2})</td>
</tr>
<tr>
<td>(E)</td>
<td>kg m s(^{-2})</td>
</tr>
<tr>
<td>(e)</td>
<td>m(^2) s(^{-2})</td>
</tr>
<tr>
<td>(\gamma)</td>
<td>1</td>
</tr>
</tbody>
</table>

\[
\frac{[\rho]}{[t]} = \frac{[\rho][u]}{[x]} \quad \text{from: } \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho u)
\]

\[
[u] = m s\(^{-1}\)
\]

\[
[p] = [\rho][u]^2 = kg m s\(^{-2}\) \quad \text{from: } \rho u^2 + p
\]

\[
[E] = [p] = kg m s\(^{-2}\) \quad \text{from: } E + p
\]

\[
[E] = [\rho][e] \quad \text{from: } E = \cdots + \rho e
\]

\[
[e] = m^2 s\(^{-2}\)
\]

\[
[\gamma] = 1 \quad \text{from: } \gamma - 1
\]
Problem 2 (15 points)

Five energy levels for a system are shown in the phase plane below. (a) List the energy levels (red, orange, yellow, green, blue, violet) in order from lowest energy to highest energy. (b) Mark all stable ("•") and unstable ("○") equilibria. (c) Sketch energy contours corresponding to all unstable equilibria (energy contours may contain more than one component; be sure to sketch them all). (d) Add arrows to all contours, including the ones you have added. (e) Sketch the potential energy function and show the energy levels corresponding to the five colored energy contours.

The energy levels are actually already in order: red, orange, yellow, green, blue, violet.
Problem 3 (15 points)

A massless wheel (radius 1) with a point mass attached off-center (by amount $a$) is free to roll (without slipping) along a flat surface. The location of the wheel is parameterized by the position $(x(t), 1)$ of the wheel’s center. When the wheel’s center is at $(0, 1)$, the mass is located at $(a, 1)$; this is the configuration pictured.

(a) What is the total potential energy ($\phi$)?
(b) What is the total kinetic energy ($KE$)?
(c) Find equations of motion for the wheel: $\ddot{x} = f(x, \dot{x})$.

(a) The polar angle the mass with respect to the wheel’s center is $\theta = -x$, so the mass is at $z = (x + a \cos x, 1 - a \sin x)$. Since a constant shift does not matter, the gravitational energy is $\phi = -mga \sin x$.

(b) The mass’s velocity is $\dot{z} = \dot{x}(1 - a \sin x, -a \cos x)$. Then, $KE = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2((1 - a \sin x)^2 + (-a \cos x)^2) = \frac{1}{2}m\dot{x}^2(a^2 + 1 - 2a \sin x)$.

(c) The equations of motion can be obtained from conservation of energy.

\[
E = KE + \phi \\
= \frac{1}{2}m\dot{x}^2(a^2 + 1 - 2a \sin x) - mga \sin x \\
0 = \dot{E} \\
= m\ddot{x}(a^2 + 1 - 2a \sin x) - am\dot{x}^2 \cos x - mga \dot{x} \cos x \\
0 = m\ddot{x}(a^2 + 1 - 2a \sin x) - am(\dot{x}^2 + g) \cos x \\
\ddot{x} = \frac{a(\dot{x}^2 + g) \cos x}{a^2 + 1 - 2a \sin x}
\]