Problem 1

The phase plane below shows several trajectories for some system. (a) Why can no energy exist for this system? (b) Add arrows to the trajectories. (c) Mark all stable (“•”) and unstable (“◦”) equilibria.

(a) If there were an energy, then the phase plane would be up-down symmetric.
Problem 2

Solve the PDE for \( z(x, t) \), subject to the initial conditions \( z(x, 0) = f(x) \).

\[
\frac{\partial z}{\partial t} + \frac{\partial}{\partial x}(xtz) = 0.
\]

Use method of characteristics.

\[
\frac{\partial z}{\partial t} + \frac{\partial}{\partial x}(xtz) = 0
\]
\[
\frac{\partial z}{\partial t} + xt \frac{\partial z}{\partial x} = -tz
\]
\[
\frac{d}{dt} z(y(t), t) = \frac{\partial z}{\partial t}(y(t), t) + \frac{\partial z}{\partial x}(y(t), t)y'(t)
\]
\[
= \frac{\partial z}{\partial t}(y(t), t) + y(t)t \frac{\partial z}{\partial x}(y(t), t) \quad y'(t) = y(t)t
\]
\[
= -tz(y(t), t)
\]
\[
y'(t) = y(t)t
\]
\[
y(t) = y_0 e^{\frac{1}{2}t^2}
\]

To evaluate \( z(x, t) \), we first need to know what characteristic \((x, t)\) is on.

\[
y'(t) = y(t)t
\]
\[
y'(t) = y(t)t
\]
\[
\ln |y(t)| = \frac{1}{2} t^2 + c_0
\]
\[
y(t) = y_0 e^{\frac{1}{2}t^2}
\]
\[
x = y_0 e^{\frac{1}{2}t^2}
\]
\[
y_0 = xe^{-\frac{1}{2}t^2}
\]

The characteristic started at \((y_0, t)\). The value of \( z \) evolves along this characteristic according to

\[
g'(t) = -tg(t) \quad g(t) = z(y(t), t)
\]
\[
g'(t) = -t \quad g(t) = -t
\]
\[
\ln |g(t)| = -\frac{1}{2} t^2 + c_0
\]
\[
g(t) = g(0)e^{-\frac{1}{2}t^2}
\]
\[
= z(y_0, 0)e^{-\frac{1}{2}t^2}
\]
\[
= f(y_0)e^{-\frac{1}{2}t^2}
\]
\[
z(x, t) = f\left(x e^{-\frac{1}{2}t^2}\right)e^{-\frac{1}{2}t^2}
\]
Problem 3

Consider the linear car-following model,

\[
\frac{d^2x_n}{dt^2}(t + T) = -\lambda \left( \frac{dx_n}{dt}(t) - \frac{dx_{n-1}}{dt}(t) \right),
\]

with a response time \(T\) (a delay). Assume all drivers behave the same. Assume the \(n\)-th driver’s velocity varies periodically

\[
v_n = \text{Re}(1 + f_n e^{i\omega t}),
\]

where \(f_n\) measures the amplification or decay which occurs. Show that this implies

\[
f_n = \left( 1 + \frac{i\omega}{\lambda} e^{i\omega T} \right)^n f_0,
\]

where \(0 < f_0 < 1\). Note: although \(f_0\) is real, \(f_n\) will in general be complex.

\[
\begin{align*}
v_{n-1}(t) &= \text{Re}(1 + f_{n-1} e^{i\omega t}) \\
v_n(t) &= \text{Re}(1 + f_n e^{i\omega t}) \\
\frac{dv_n}{dt}(t) &= \text{Re}(i\omega f_n e^{i\omega t}) \\
\frac{dv_n}{dt}(t + T) &= \text{Re}(i\omega f_n e^{i\omega (t+T)}) \\
&= \text{Re}(i\omega f_n e^{i\omega T} e^{i\omega t}) \\
\frac{dv_n}{dt}(t + T) &= -\lambda(v_n(t) - v_{n-1}(t))
\end{align*}
\]

\[
\text{Re}(i\omega f_n e^{i\omega T} e^{i\omega t}) = -\lambda(\text{Re}(1 + f_n e^{i\omega t}) - \text{Re}(1 + f_{n-1} e^{i\omega t}))
\]

This must be true for any \(t\). The factor of \(e^{i\omega t}\) will mix the real and complex components of the factor in brackets, so for the real part to be zero for any \(t\), the factor in brackets must also be zero.

\[
0 = i\omega f_n e^{i\omega T} + \lambda(f_n - f_{n-1})
\]

\[
i\omega f_n e^{i\omega T} + \lambda f_n = \lambda f_{n-1}
\]

\[
(i\omega e^{i\omega T} + \lambda)f_n = \lambda f_{n-1}
\]

\[
f_n = \frac{\lambda}{i\omega e^{i\omega T} + \lambda} f_{n-1}
\]

\[
f_n = \left( \frac{\lambda}{i\omega e^{i\omega T} + \lambda} \right) f_0 = \left( 1 + \frac{i\omega}{\lambda} e^{i\omega T} \right)^n f_0
\]
Problem 4

An object moves in the oscillating velocity field \( \mathbf{u}(\langle x, y \rangle, t) = \langle x \cos t, -y \cos t \rangle \). Determine the path of the particle in this velocity field that started at \((x_0, y_0)\).

From the usual relation \( \mathbf{x}(t) = \mathbf{u}(\mathbf{x}(t), t) \)

\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} = \begin{pmatrix}
x \cos t \\
-y \cos t
\end{pmatrix}
\]

\[
\dot{x} = \cos t
\]

\[
\ln |x| = \sin t + c_0
\]

\[
x = c_1 e^{\sin t}
\]

\[
x_0 = c_1 e^{\sin 0} = c_1
\]

\[
\dot{y} = -\cos t
\]

\[
\ln |y| = -\sin t + c_2
\]

\[
y = c_3 e^{-\sin t}
\]

\[
y_0 = c_3 e^{-\sin 0} = c_3
\]

\[
\begin{pmatrix} x \\
y \end{pmatrix} = \begin{pmatrix} x_0 e^{\sin t} \\
y_0 e^{-\sin t} \end{pmatrix}
\]
Problem 5

An undamped linear spring system \( m \ddot{x} + kx = 0 \) is entirely described by its mass \( m \) and stiffness \( k \). In a homework problem, you determined that the most general expression with units of time that could be derived for this system was \( T = a \sqrt{\frac{m}{k}} \), where \( a \) is a unitless constant. The damped linear spring system \( m \ddot{x} + c \dot{x} + kx = 0 \) is entirely described by its mass \( m \), stiffness \( k \), and damping coefficient \( c \). Find three independent expressions that have units of time that could be derived for this damped system. (The expressions are considered to be not independent if one of them can be obtained by adding constant multiples of the other two.) Introducing constants with units is not allowed. The quantities \( x, \dot{x}, \) and \( \ddot{x} \) are not parameters and cannot be used.

The reason the homework solution was so simple is that it was not possible to derive a unitless quantity in a nontrivial way. Because of this, it was not possible to introduce more complicated operations (e.g., trigonometry) into the expression. Here, things are different. Since \([m] = \text{kg}, [c] = \text{kg s}^{-1}, \) and \([k] = \text{kg s}^{-2}\), the quantity \( A = \frac{mk}{c^2} \) is unitless. A dependence like \( T = \cos \left( \frac{mk}{c^2} \right) \sqrt{\frac{m}{k}} \) is possible. In fact, any real-valued function could be used. The most general expression with units of \( \text{kg}^n \text{s}^p \) is

\[
Q_{n,p} = f \left( \frac{mk}{c^2} \right) m^n c^{-p},
\]

where \( f(x) \) is some real-valued function. That this is the most general form that can be derived can be seen since (a) the quantities with units (\( m, c, \) and \( k \)) are of this form, (b) this form is closed under the available operations:

- addition, subtraction, multiplication, division
- powers
- scaling by unitless constants
- taking functions of unitless expressions

The solution is thus

\[
T = Q_{0,1} = f \left( \frac{mk}{c^2} \right) \frac{m}{c}.
\]

There are a lot of ways to get three independent ones, such as \( f(x) = 1, f(x) = x, \) and \( f(x) = x^2. \)
Problem 6

A mass hangs from a rope, which is wrapped around a pulley (negligible radius) and connected to a winch. Let \( \ell(t) \) be the length of rope between the pulley and the mass. A motor in the winch lowers the mass by letting out rope at a constant rate of \( \dot{\ell} = a \). Let \( \ell_0 = \ell(0) \). The mass is free to swing back and forth from the rope. Let \( \theta(t) \) be the angle of the mass with respect to vertical. Find equations of motion for the angle \( \theta(t) \). Is energy conserved in this system? Why?

Energy is not conserved; the mass does work on the motor by falling. This is most obvious if \( \theta(0) = 0 \), in which case the mass is falling at a rate of \( a \). Since energy is not conserved, this problem cannot be worked out using conservation of energy. The rope’s length is: \( \ell = \ell_0 + at \).

\[
\begin{align*}
n &= \begin{pmatrix} \sin \theta \\ -\cos \theta \end{pmatrix} \\
t &= \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \\
\dot{n} &= \dot{\theta} t \\
t &= -\dot{n} \\
x &= \ell n \\
\ddot{x} &= \dot{\ell} n + \ell \dot{n} \\
&= a n + \ell \dot{n} \\
\ddot{x} &= a \dot{n} + \ell \ddot{n} \dot{t} + \ell n \ddot{t} \\
&= 2a \dot{n} + \ell \ddot{n} \dot{t} - \ell \dot{n}^2 n \\
T &= -T n \\
g &= \begin{pmatrix} 0 \\ -mg \end{pmatrix} \\
m \ddot{x} &= T + g \\
2am\dot{\theta} t + \ell m \dot{\theta} t - \ell m \dot{\theta}^2 n &= -T n - mg j \\
2am\dot{\theta} \cdot t + \ell m \dot{\theta} \cdot t - \ell m \dot{\theta}^2 n \cdot t &= -T n \cdot t - mg \cdot t \\
2am\dot{\theta} + \ell m \ddot{\theta} &= -mg \sin \theta \\
2a\ddot{\theta} + \ell \ddot{\theta} &= -g \sin \theta \\
\ell \ddot{\theta} + 2a \ddot{\theta} + g \sin \theta &= 0 \\
(\ell_0 + at) \ddot{\theta} + 2a \ddot{\theta} + g \sin \theta &= 0
\end{align*}
\]
Problem 7

For simplicity, let \( \rho_{\text{max}} = 1 \) and \( u_{\text{max}} = 1 \). Then, \( \dot{\rho}(\rho) = 1 - \rho \). The periodic initial density profile is shown below. One period \( (0 < x < 2) \) of the initial density is

\[
\rho(x, 0) = \begin{cases} 
\frac{x}{2} & 0 < x < 1 \\
1 - \frac{x}{2} & 1 < x < 2
\end{cases}
\]

Determine the time \( T \) of first shock and find \( \rho(x, t) \) for all \( x \) and all \( t \in [0, T) \).

There are initially no discontinuities, so there are initially no rarefaction or shocks to deal with. Since places with increasing density exist, shocks will form later. Lets first follow a characteristic from \( x_0 \in (0, 1) \).

\[
\rho = \frac{x_0}{2} \\
c = \dot{q}'(\rho) = 1 - 2\rho = 1 - x_0 \\
x = x_0 + ct = x_0 + (1 - x_0)t \\
x_0 = \frac{x - t}{1 - t} \\
\rho = \frac{x_0}{2} = \frac{x - t}{2(1 - t)}
\]

Note that infinite slope occurs at \( t = 1 \), suggesting the formation of a shock at this time. Next, \( x_0 \in (1, 2) \).

\[
\rho = 1 - \frac{x_0}{2} \\
c = \dot{q}'(\rho) = 1 - 2\rho = x_0 - 1 \\
x = x_0 + ct = x_0 + (x_0 - 1)t \\
x_0 = \frac{x + t}{1 + t} \\
\rho = \frac{1 - x_0}{2} = 1 - \frac{x + t}{2(1 + t)} = \frac{2 + t - x}{2(1 + t)}
\]

We can determine which region we are in by looking at the characteristics separating them, which start from \( x_0 = 0 \) \( (\rho = 0, c = 1, x = x_0 + ct = t) \) and \( x_0 = 1 \) \( (\rho = 1, c = 0, x = x_0 + ct = 1) \). Note that these also meet at \( t = 1 \), which is also the shock time that is produced from the normal time-of-first-shock derivation. For \( 0 \leq t \leq 1 \), one period of the solution is given by

\[
\rho(x, t) = \begin{cases} 
\frac{x-t}{2(1-t)} & t < x < 1 \\
\frac{2+t-x}{2(1+t)} & 1 < x < 2 + t
\end{cases}
\]
Problem 8

A road has an offramp in the interval $[0, a]$ over which cars exit the road at a rate proportional to the number of cars. That is, if the density of cars in some subinterval $[x_0, x_0 + \Delta x] \subset [0, a]$ is $\rho$, then over time $\Delta t$, $\beta \rho \Delta x \Delta t$ cars will exit the road. (a) Derive a PDE that describes the evolution of the traffic density over time. (b) Assuming $\hat{u}(\rho) = u_{\text{max}} \left(1 - \frac{\rho}{\rho_{\text{max}}}\right)$ and $\rho(x, 0) = \rho_0$, find an equation for the characteristic that starts at $x_0$.

\[
\frac{dN}{dt} = \frac{d}{dt} \int_a^b \rho(x, t) \, dx = q(a, t) - q(b, t) - \int_a^b \beta \rho(x, t) \, dx
\]

\[
\int_a^x \frac{\partial \rho}{\partial t}(\hat{x}, t) \, d\hat{x} = q(a, t) - q(x, t) - \int_a^x \beta \rho(\hat{x}, t) \, d\hat{x}
\]

\[
\frac{\partial}{\partial x} \left( \int_a^x \frac{\partial \rho}{\partial t}(\hat{x}, t) \, d\hat{x} \right) = \frac{\partial}{\partial x} \left( q(a, t) - q(x, t) - \int_a^x \beta \rho(\hat{x}, t) \, d\hat{x} \right)
\]

\[
\frac{\partial \rho}{\partial t}(x, t) + \frac{\partial q}{\partial x}(x, t) = -\beta \rho(x, t)
\]

Now we need to get the characteristic. Let $z(t)$ be this characteristic, with $z(0) = x_0$.

\[
\frac{d}{dt} \rho(z(t), t) = \frac{\partial \rho}{\partial t}(z(t), t) + \frac{dz}{dt}(t) \frac{\partial \rho}{\partial x}(z(t), t)
\]

\[
= \frac{\partial \rho}{\partial t}(z(t), t) + \frac{d\hat{q}}{d\rho}(\rho(z(t), t)) \frac{\partial \rho}{\partial x}(z(t), t)
\]

\[
\frac{dz}{dt}(t) = \frac{d\hat{q}}{d\rho}(\rho(z(t), t))
\]

\[
\rho(z(t), t) = \rho_0 e^{-\beta t}
\]

\[
\frac{d\hat{q}}{d\rho}(\rho(z(t), t)) = \frac{d\hat{q}}{d\rho}(\rho_0 e^{-\beta t})
\]

\[
= u_{\text{max}} \left(1 - \frac{2\rho_0 e^{-\beta t}}{\rho_{\text{max}}}\right)
\]

\[
z(t) = k + u_{\text{max}} t + \frac{2\rho_0 u_{\text{max}}}{\beta \rho_{\text{max}}} (e^{-\beta t} - 1)
\]

\[
z(t) = x_0 + u_{\text{max}} t + \frac{2\rho_0 u_{\text{max}}}{\beta \rho_{\text{max}}} (e^{-\beta t} - 1)
\]
Problem 9

A massless wheel (radius 1) with a point mass attached off-center (by amount $a$) is free to roll (without slipping) along a flat surface. The location of the wheel is parameterized by the position $(x(t), 1)$ of the wheel's center. When the wheel's center is at $(0, 1)$, the mass is located at $(a, 1)$; this is the configuration pictured. What is the total kinetic energy?

The polar angle the mass with respect to the wheel’s center is $\theta = -x$, so the mass is at $z = (x + a \cos x, 1 - a \sin x)$. The mass’s velocity is $\dot{z} = \dot{x}(1 - a \sin x, -a \cos x)$. Then, $KE = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x}^2((1 - a \sin x)^2 + (-a \cos x)^2) = \frac{1}{2}m\dot{x}^2(a^2 + 1 - 2a \sin x)$. 

Problem 10

A long road has an initial uniform traffic density $\rho(x, 0) = \frac{\rho_{\text{max}}}{5}$. At $t = 0$, a traffic accident occurs at $x = 0$, which effectively limits the flow rate past $x = 0$ to $q(0, t) \leq \frac{3}{16} u_{\text{max}} \rho_{\text{max}}$. Determine the traffic density for $t > 0$. Assume $\dot{u}(\rho) = u_{\text{max}} \left(1 - \frac{\rho}{\rho_{\text{max}}} \right)$. Hint: this problem should not take long to work out.

While this problem looks extremely similar to one seen earlier in the course, it is in fact far simpler. The flow of traffic under initial traffic conditions is

\[
q(\rho) = \rho u_{\text{max}} \left(1 - \frac{\rho}{\rho_{\text{max}}} \right)
\]

\[
q(\rho(x, 0)) = \frac{\rho_{\text{max}}}{5} u_{\text{max}} \left(1 - \frac{\frac{\rho_{\text{max}}}{5}}{\rho_{\text{max}}} \right)
\]

\[
= \frac{4}{25} u_{\text{max}} \rho_{\text{max}}
\]

\[
< \frac{3}{16} u_{\text{max}} \rho_{\text{max}}
\]

Since the initial traffic flow is within the flow limits imposed by the accident, it is unaffected. The traffic density remains $\rho(x, t) = \frac{\rho_{\text{max}}}{5}$. 