Problem 1

A long road has an initial uniform traffic density \( \rho(x, 0) = \frac{\rho_{\text{max}}}{3} \). At \( t = 0 \), a traffic accident occurs at \( x = 0 \), which effectively limits the flow rate past \( x = 0 \) to \( q(0, t) = \frac{3}{16} u_{\text{max}} \rho_{\text{max}} \). Determine the traffic density for \( t > 0 \). Assume \( \dot{u}(\rho) = u_{\text{max}} \left( 1 - \frac{\rho}{\rho_{\text{max}}} \right) \).

This is essentially the same problem as the light turning red. The only difference is the flow rate. First, we find the densities that correspond to the flux for the accident location.

\[
q = \rho u_{\text{max}} \left( 1 - \frac{\rho}{\rho_{\text{max}}} \right)
\]

\[
\rho u_{\text{max}} \left( 1 - \frac{\rho}{\rho_{\text{max}}} \right) = \frac{3}{16} u_{\text{max}} \rho_{\text{max}}
\]

\[
\rho^2 - \rho \rho_{\text{max}} + \frac{3}{16} \rho_{\text{max}}^2 = 0
\]

\[
\left( \rho - \frac{1}{4} \rho_{\text{max}} \right) \left( \rho - \frac{3}{4} \rho_{\text{max}} \right) = 0
\]

\[
\rho = \frac{1}{4} \rho_{\text{max}}, \frac{3}{4} \rho_{\text{max}}
\]

Of these, \( \rho = \frac{1}{4} \rho_{\text{max}} \) corresponds to a forward moving characteristic and \( \rho = \frac{3}{4} \rho_{\text{max}} \) corresponds to a backward moving characteristic. Thus, I expect \( \rho = \frac{1}{4} \rho_{\text{max}} \) in front of the accident and \( \rho = \frac{3}{4} \rho_{\text{max}} \) behind it.

Label the regions \( \rho_r \), \( \rho_b \), and \( \rho_g \) (red, green, blue).

\[
\rho_r = \frac{\rho_{\text{max}}}{3}
\]

\[
q_r = \rho_r u_{\text{max}} \left( 1 - \frac{\rho_r}{\rho_{\text{max}}} \right) = \frac{2}{9} \rho_{\text{max}} u_{\text{max}}
\]

\[
\rho_g = \frac{3\rho_{\text{max}}}{4}
\]

\[
q_g = \frac{3}{16} \rho_{\text{max}} u_{\text{max}}
\]

\[
\rho_b = \frac{\rho_{\text{max}}}{4}
\]

\[
q_b = \frac{3}{16} \rho_{\text{max}} u_{\text{max}}
\]
Next, we compute the shock speeds for the red-green and blue-red shocks.

\[
\frac{ds_{rg}}{dt} = \frac{q_r - q_g}{\rho_r - \rho_g} = \frac{2}{\frac{2}{3} \rho_{\text{max}} u_{\text{max}}} - \frac{3}{16} \rho_{\text{max}} u_{\text{max}} = -\frac{1}{12} u_{\text{max}}
\]

\[
\frac{ds_{br}}{dt} = \frac{q_b - q_r}{\rho_b - \rho_r} = \frac{3}{16} \rho_{\text{max}} u_{\text{max}} - \frac{2}{9} \rho_{\text{max}} u_{\text{max}} = \frac{5}{12} u_{\text{max}}
\]

Finally, we can assemble the solution.

\[
\rho(x, t) = \begin{cases} \\
\frac{\rho_{\text{max}}}{3} & x < -\frac{1}{12} u_{\text{max}} t \\
\frac{3\rho_{\text{max}}}{4} & -\frac{1}{12} u_{\text{max}} t < x < 0 \\
\frac{\rho_{\text{max}}}{4} & 0 < x < \frac{5}{12} u_{\text{max}} t \\
\frac{5\rho_{\text{max}}}{3} & \frac{5}{12} u_{\text{max}} t < x \\
\end{cases}
\]