Problem 1

Each of the derivations below is incorrect. Find the mistakes.

(a) Problem: Let \( f(x, y) = 2x^2 + y^3 \) and \( g(x, y) = f(x, y) + f(y, x) \). Compute \( g_x(1, 2) \).

Solution 1:

\[
\begin{align*}
  f(x, y) &= 2x^2 + y^3 \\
  f_x(x, y) &= 4x \\
  f_x(1, 2) &= 4 \\
  g(x, y) &= f(x, y) + f(y, x) \\
  g_x(x, y) &= f_x(x, y) + f_x(y, x) \\
  g_x(1, 2) &= f_x(1, 2) + f_x(2, 1) \\
  &= 4 + 8 = 12
\end{align*}
\]

Solution 2:

\[
\begin{align*}
  g(x, y) &= f(x, y) + f(y, x) \\
  &= (2x^2 + y^3) + (2y^2 + x^3) \\
  &= 2x^2 + x^3 + 2y^2 + y^3 \\
  g_x(x, y) &= 4x + 3y^2 \\
  g_x(1, 2) &= 4 + 3 = 7
\end{align*}
\]

(b) Let \( K = \frac{1}{2}mv^2 \) be the kinetic energy of a particle. Define the quantity \( A = \frac{\partial^2 K}{\partial m^2} \). In some contexts, velocity is an inconvenient variable, and momentum \( p = mv \) is preferred instead. In such cases, one would write \( K = \frac{p^2}{2m} \). Compute \( A \) if \( m = 2 \) and \( p = 4 \).

Solution 1: From the momentum form of \( K \), \( A = \frac{\partial^2 K}{\partial m^2} = \frac{p^2}{m^3} = \frac{16}{8} = 2 \).

Solution 2: From the velocity form of \( K \), it is clear that \( A = 0 \).

(c) Let \( K = \frac{1}{2}m\dot{x}^2 \) be the kinetic energy of a particle that is falling from rest under gravity (\( \ddot{x} = -g \)). Does \( K \) depend on time?

Solution 1: No; \( \frac{\partial K}{\partial t} = 0 \), so \( K \) does not depend on time.

Solution 2: Yes; \( \frac{dK}{dt} = m\dot{x}\ddot{x} \neq 0 \), so \( K \) does depend on time.

Problem 2

A disk spins about the origin, as shown in the diagram.

(a) Let \( P \) be a dot painted on the disk, initially at location \((x_0, y_0)\). Assuming the angular velocity \( \dot{\theta} \) is constant and that the disk makes one complete revolution after time \( T \), find the velocity and position of \( P \) at an arbitrary time \( t \).
(b) Under the assumptions of part (a), find the velocity field \( \vec{u}(\vec{x}, t) \).

(c) Does the velocity change with time?

(d) Assume instead that the angular velocity decays exponentially. That is, \( \ddot{\theta} = -r \dot{\theta} \). The disk completes its first rotation at time \( T \), by which time its angular velocity has halved. What is \( r \)? Find the velocity and position of \( P \) at an arbitrary time \( t \).

(e) Under the assumptions of part (c), find the velocity field \( \vec{u}(\vec{x}, t) \).