Problem 1

A wheel with radius $r$ spins freely and without friction about its center. One end of a spring is attached to the wheel at a distance of $s$ from the center of the wheel. The other end of the spring (restlength $\ell$, spring constant $k$) is fixed to a point $d$ below the wheel’s center. A mass $m$ is attached to the wheel opposing the spring’s attachment point. Find the equations of motion for the wheel, parameterized by the polar angle $\theta$ of the spring’s attachment point. Your answer should be an ODE of the general form $\ddot{\theta} = f(\theta, \dot{\theta}, t)$.

Hint: formulating energy first is easier.

Let $x = s \left( \frac{\cos \theta}{\sin \theta} \right)$ be the attachment location. The mass is at $y = -x$, so its kinetic energy is $KE = \frac{1}{2}m||\dot{y}||^2 = \frac{1}{2}ms^2\dot{\theta}^2$. The potential of the mass is $\theta_1 = mgh = -mgs\sin \theta$. Next, I need the potential energy for the spring.

$$L = \sqrt{(s \cos \theta)^2 + (s \sin \theta + d)^2} = \sqrt{s^2 + d^2 + 2sd\sin \theta}$$

$$\phi_2 = \frac{1}{2}k(L - \ell)^2 = \frac{1}{2}k\left( \sqrt{s^2 + d^2 + 2sd\sin \theta} - \ell \right)^2$$

I can now assemble these to get the energy and use conservation of energy to get the ODE.

$$E = \frac{1}{2}ms^2\dot{\theta}^2 - mgs \sin \theta + \frac{1}{2}k\left( \sqrt{s^2 + d^2 + 2sd\sin \theta} - \ell \right)^2$$

$$0 = \dot{E}$$

$$= ms^2\ddot{\theta} - mgs \dot{\theta} \cos \theta + \frac{k}{2\sqrt{s^2 + d^2 + 2sd\sin \theta}}\left(2sd \cos \theta\right)\dot{\theta}$$

$$= ms^2\ddot{\theta} - mgs \dot{\theta} \cos \theta + ksd \cos \theta \left(1 - \frac{\ell}{\sqrt{s^2 + d^2 + 2sd\sin \theta}}\right)\dot{\theta}$$

$$0 = ms^2\ddot{\theta} - mgs \dot{\theta} \cos \theta + ksd \cos \theta \left(1 - \frac{\ell}{\sqrt{s^2 + d^2 + 2sd\sin \theta}}\right)$$

$$ms^2\ddot{\theta} = mgs \cos \theta - ksd \cos \theta \left(1 - \frac{\ell}{\sqrt{s^2 + d^2 + 2sd\sin \theta}}\right)$$

$$\ddot{\theta} = \frac{g}{s} \cos \theta - \frac{kd}{ms} \cos \theta \left(1 - \frac{\ell}{\sqrt{s^2 + d^2 + 2sd\sin \theta}}\right)$$