Problem 1
This group work activity is based around exploring the second order ordinary differential equation
\[ tx'' + x' + tx = \frac{1}{4} \sin t. \]
Write this second order ODE as a first order ODE of the form \( z' = f(z, t) \) by introducing a new variable \( v = x' \). Note that \( z \) in our case will need to be the vector \( \langle x, v \rangle \). Find \( f(z, t) \).

\[
\begin{align*}
x' &= v \\
tx'' + x' + tx &= \frac{1}{4} \sin t \\
tv' + v + tx &= \frac{1}{4} \sin t \\
tv' &= \frac{1}{4} \sin t - v - tx \\
v' &= \frac{\sin t}{4t} \frac{v}{t} - x \\
f(\langle x, v \rangle, t) &= \left\langle v, \frac{\sin t}{4t} \frac{v}{t} - x \right\rangle
\end{align*}
\]

Problem 2
Write a Matlab function which takes two arguments and computes \( f(z, t) \) (note that the argument \( z \) and the result will be vectors). You are not required to use Matlab; you may use another language if you prefer. You will need to be able to generate line plots.

Problem 3
Plot the vector valued function \( f((\cos t, -\sin t), t) \) for \( t \in [0, 4\pi] \) using the Matlab function you wrote. Plot the function at around hundred or so sample points. (Note that you will have difficulty at \( t = 0 \); one simple solution is to plot it at a tiny value like \( t = 10^{-10} \) instead.)
Problem 4

When solving ODE's numerically, we compute approximate values for $z_n \approx z(t_n)$ at a large number $N$ of sample times $t_0 \ldots t_N$ in the interval $[0, T]$. The sample points are equally spaced ($t_{n+1} = t_n + h$ for fixed $h$), with $t_0 = 0$ and $t_N = T$. The forward Euler method for approximating the solution to an ODE $z' = f(z, t)$ is given by

$$
\begin{align*}
z_0 &= \langle x(0), x'(0) \rangle \\
z_{n+1} &= z_n + hf(t_n, z_n)
\end{align*}
$$

Compute $z_0 \ldots z_N$ for $N = 400$, $T = 20$, $x(0) = 1$. (What is $x'(0)$?)

Plugging $t = 0$ into the ODE gives us $x' = 0$. Thus, $z_0 = \langle 1, 0 \rangle$.

Problem 5

Generate a plot of the points $(t_n, x(t_n))$. 

| 2 |
Problem 6

Approximate the maximum and minimum values of \( x(t) \) over the interval \([0, T]\).

The solution, computed numerically as above is approximately \(-0.37\) for the min and \(1.00\) for the max.