Math 142-2, Homework 5

Your name here

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Problem 57.2

57.2. Suppose a velocity field is given:

$$u(x,t) = \frac{30x + 30L}{15t + L}$$

(a) Determine the motion of a car which starts at $x = L/2$ at $t = 0$. [Hint: Why does $dx/dt = (30x + 30L)/(15t + L)$? Solve this differential equation. It is separable.]

Your solution goes here

(b) Show that $u(x,t)$ is constant along straight lines in the $x - t$ plane, but the car does not move at a constant velocity.

Your solution goes here

Problem 57.3

Suppose that the velocity field $u(x,t)$ is known. What mathematical problem needs to be solved in order to determine the position of a car at later times, which starts (at $t = 0$) at $x = L$?

Your solution goes here

Problem 57.5

Suppose that $u(x,t) = e^{-t}$.

(a) Sketch curves in $x - t$ space along which $u(x,t)$ is constant.

Your solution goes here

(b) Determine the time dependence of the position of any car.

Your solution goes here

(c) In the same $x - t$ space used in part (a), sketch various different car paths.

Your solution goes here

Problem 57.6

Consider an infinite number of cars, each designated by a number β. Assume the car labeled β starts from $x = \beta$ ($\beta > 0$) with zero velocity, and also assume it has a constant acceleration β.

(a) Determine the position and velocity of each car as a function of time.

Your solution goes here

(b) Sketch the path of a typical car.

Your solution goes here

(c) Determine the velocity field $u(x, t)$.

Your solution goes here

(d) Sketch curves along which $u(x, t)$ is a constant.

Your solution goes here

Problem 59.2

Suppose that at position x_0 the traffic flow is known, $q(x_0, t)$, and varies with time. Calculate the number of cars that pass x_0, between $t = 0$ and $t = t_0$.

Your solution goes here

Problem 59.3

In an experiment the total number of cars that pass a position x_0 after $t = 0$, $M(x_0, t)$, is measured as a function of time. Assume this series of points has been smoothed to make a continuous curve.

(a) Briefly explain why the curve $M(x_0, t)$ is increasing as t increases.

Your solution goes here

(b) What is the traffic flow at $t = \tau$?

Your solution goes here

Problem 60.1

Consider a semi-infinite highway $0 \le x < \infty$ (with no entrances or exits other than at $x = 0$). Show that the number of cars on the highway at time t is

$$N_0 + \int_0^t q(0, \tau)\, d\tau,$$

where N_0 is the number of cars on the highway at $t = 0$. (You may assume that $\rho(x, t) \to 0$ as $x \to \infty$.)

Your solution goes here

Problem 60.2

Suppose that we are interested in the change in the number of cars $N(t)$ between two observers, one fixed at $x = a$ and the other moving in some prescribed manner, $x = b(t)$:

$$N(t) = \int_a^{b(t)} \rho(x, t)\, dx$$

(a) The derivative of an integral with a variable limit is

$$\frac{dN}{dt} = \frac{db}{dt}\rho(b, t) + \int_a^{b(t)} \frac{\partial \rho}{\partial t}\, dx.$$

(Note that the integrand, $\rho(x, t)$, also depends on t.) Show this result either by considering $\lim_{\Delta t \to 0}[N(t + \Delta t) - N(t)]/\Delta t$ or by using the chain rule for derivatives.

Your solution goes here

(b) Using $\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}(\rho u)$ show that

$$\frac{dN}{dt} = -\rho(b,t)\left(u(b,t) - \frac{db}{dt}\right) + \rho(a,t)u(a,t).$$

Your solution goes here

(c) Interpret the result of part (b) if the moving observer is in a car moving with the traffic.

Your solution goes here

Problem 61.3

Assume that a velocity field, $u(x,t)$, exists. Show that the acceleration of an individual car is given by

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}.$$

Your solution goes here

Problem 63.6

Assume that $\hat{u} = \hat{u}(\rho)$. If α equals a car's acceleration, show that

$$\alpha = -\rho\frac{d\hat{u}}{d\rho}\frac{\partial u}{\partial x}.$$

Is the minus sign reasonable? Note that I have given $\hat{u}$ and u different names to make their different functional dependence explicit. In particular, $u(x(t),t) = \hat{u}(\rho(x(t),t))$.

Your solution goes here

Problem 63.7

Consider exercise 61.3. Suppose that drivers accelerate such that

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{a^2}{\rho}\frac{\partial \rho}{\partial x},$$

where a is a positive constant.

(a) Physically interpret this equation.

Your solution goes here

(b) If u only depends on ρ and the equation of conservation of cars is valid, show that

$$\frac{du}{d\rho} = -\frac{a}{\rho}.$$

Note: You may assume $\frac{d\hat{u}}{d\rho} < 0$.

Your solution goes here

(c) Solve the differential equation in part (b), subject to the condition that $u(\rho_{\max}) = 0$. The resulting flow-density curve fits quite well to the Lincoln Tunnel data.

Your solution goes here

(d) Show that a is the velocity which corresponds to the road's capacity.

Your solution goes here

(e) Discuss objections to this theory for small densities.

Your solution goes here