Math 142-2, Homework 5

Your name here

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Problem 57.2

57.2. Suppose a velocity field is given:

\[ u(x, t) = \frac{30x + 30L}{15t + L} \]

(a) Determine the motion of a car which starts at \( x = L/2 \) at \( t = 0 \). [Hint: Why does \( dx/dt = (30x + 30L)/(15t + L) \)? Solve this differential equation. It is separable.]

Your solution goes here

(b) Show that \( u(x, t) \) is constant along straight lines in the \( x-t \) plane, but the car does not move at a constant velocity.

Your solution goes here

Problem 57.3

Suppose that the velocity field \( u(x, t) \) is known. What mathematical problem needs to be solved in order to determine the position of a car at later times, which starts (at \( t = 0 \)) at \( x = L \)?

Your solution goes here

Problem 57.5

Suppose that \( u(x, t) = e^{-t} \).

(a) Sketch curves in \( x-t \) space along which \( u(x, t) \) is constant.

Your solution goes here
(b) Determine the time dependence of the position of any car.

Your solution goes here

(c) In the same x–t space used in part (a), sketch various different car paths.

Your solution goes here

Problem 57.6
Consider an infinite number of cars, each designated by a number \( \beta \). Assume the car labeled \( \beta \) starts from \( x = \beta \) (\( \beta > 0 \)) with zero velocity, and also assume it has a constant acceleration \( \dot{\beta} \).

(a) Determine the position and velocity of each car as a function of time.

Your solution goes here

(b) Sketch the path of a typical car.

Your solution goes here

(c) Determine the velocity field \( u(x, t) \).

Your solution goes here

(d) Sketch curves along which \( u(x, t) \) is a constant.

Your solution goes here

Problem 59.2
Suppose that at position \( x_0 \) the traffic flow is known, \( q(x_0, t) \), and varies with time. Calculate the number of cars that pass \( x_0 \), between \( t = 0 \) and \( t = t_0 \).

Your solution goes here
Problem 59.3

In an experiment the total number of cars that pass a position \( x_0 \) after \( t = 0 \), \( M(x_0, t) \), is measured as a function of time. Assume this series of points has been smoothed to make a continuous curve.

(a) Briefly explain why the curve \( M(x_0, t) \) is increasing as \( t \) increases.

Your solution goes here

(b) What is the traffic flow at \( t = \tau \)?

Your solution goes here

Problem 60.1

Consider a semi-infinite highway \( 0 \leq x < \infty \) (with no entrances or exits other than at \( x = 0 \)). Show that the number of cars on the highway at time \( t \) is

\[
N_0 + \int_0^t q(0, \tau) \, d\tau,
\]

where \( N_0 \) is the number of cars on the highway at \( t = 0 \). (You may assume that \( \rho(x, t) \to 0 \) as \( x \to \infty \).)

Your solution goes here

Problem 60.2

Suppose that we are interested in the change in the number of cars \( N(t) \) between two observers, one fixed at \( x = a \) and the other moving in some prescribed manner, \( x = b(t) \):

\[
N(t) = \int_a^{b(t)} \rho(x, t) \, dx
\]

(a) The derivative of an integral with a variable limit is

\[
\frac{dN}{dt} = \frac{db}{dt} \rho(b, t) + \int_a^{b(t)} \frac{\partial \rho}{\partial t} \, dx.
\]

(Note that the integrand, \( \rho(x, t) \), also depends on \( t \).) Show this result either by considering \( \lim_{\Delta t \to 0} [N(t + \Delta t) - N(t)]/\Delta t \) or by using the chain rule for derivatives.

Your solution goes here
(b) Using \( \frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}(\rho u) \) show that

\[
\frac{dN}{dt} = -\rho(b,t) \left( u(b,t) - \frac{db}{dt} \right) + \rho(a,t)u(a,t).
\]

Your solution goes here

(c) Interpret the result of part (b) if the moving observer is in a car moving with the traffic.

Your solution goes here

Problem 61.3

Assume that a velocity field, \( u(x,t) \), exists. Show that the acceleration of an individual car is given by

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x}.
\]

Your solution goes here

Problem 63.6

Assume that \( \ddot{u} = \dot{u}(\rho) \). If \( \alpha \) equals a car’s acceleration, show that

\[
\alpha = -\rho \frac{d\dot{u}}{d\rho} \frac{\partial u}{\partial x}.
\]

Is the minus sign reasonable? Note that I have given \( \ddot{u} \) and \( u \) different names to make their different functional dependence explicit. In particular, \( u(x(t),t) = \dot{u}(\rho(x(t),t)) \).

Your solution goes here

Problem 63.7

Consider exercise 61.3. Suppose that drivers accelerate such that

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -\frac{\alpha^2}{\rho} \frac{\partial \rho}{\partial x},
\]

where \( \alpha \) is a positive constant.

(a) Physically interpret this equation.
(b) If \( u \) only depends on \( \rho \) and the equation of conservation of cars is valid, show that

\[
\frac{du}{d\rho} = -\frac{a}{\rho}.
\]

Note: You may assume \( \frac{du}{d\rho} < 0 \).

(c) Solve the differential equation in part (b), subject to the condition that \( u(\rho_{\text{max}}) = 0 \). The resulting flow-density curve fits quite well to the Lincoln Tunnel data.

(d) Show that \( a \) is the velocity which corresponds to the road’s capacity.

(e) Discuss objections to this theory for small densities.