Problem 1 (5 points)
Show that if \( f(x) \) and \( g(x) \) are Lipschitz continuous, then \( h(x) = f(g(x)) \) is also Lipschitz continuous.

\[
|h(x_0) - h(x_1)| = |f(g(x_0)) - f(g(x_1))| \\
\leq K_f |g(x_0) - g(x_1)| \\
\leq K_f K_g |x_0 - x_1|
\]
Problem 2 (5 points)

For each ODE in the table below, determine whether the local and global existence and uniqueness theorems apply. If the theorem does apply, indicate a suitable region (for example, \([-1, 1] \times [-1, 1]\)) on which the theorem applies in the space in the table. If the theorem does not apply, indicate briefly why (for example, “not continuous”). There are ten spots in the table, each worth half a point. You do not need to show that your answers are correct. Only answers in the table count.

<table>
<thead>
<tr>
<th>ODE</th>
<th>Local</th>
<th>Global</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) (y' = \tan(xy)) (y(0) = \frac{\pi}{2})</td>
<td>([-\frac{1}{2}, \frac{1}{2}] \times [-2, 2])</td>
<td>Fails Lipschitz and continuity</td>
</tr>
<tr>
<td>(b) (y' = \sin(xy)) (y(0) = \frac{\pi}{2})</td>
<td>([-2, 2] \times [-2, 2])</td>
<td>([-2, 2] \times (-\infty, \infty))</td>
</tr>
<tr>
<td>(c) ((x^2 + y^2)y' = xy) (y(0) = 0)</td>
<td>Fails Lipschitz and continuity</td>
<td>Fails Lipschitz and continuity</td>
</tr>
<tr>
<td>(d) (y' = x\sin(\frac{1}{x})y) (y(0) = 0)</td>
<td>([-2, 2] \times [-2, 2])</td>
<td>([-2, 2] \times (-\infty, \infty))</td>
</tr>
<tr>
<td>(e) (y' = y\sin(\frac{1}{y})x) (y(0) = 0)</td>
<td>Fails Lipschitz</td>
<td>Fails Lipschitz</td>
</tr>
</tbody>
</table>

Note: when we say \(g(x) = x\sin(\frac{1}{x})\), we implicitly assume that \(g(0) = 0\). The function \(g(x)\) is continuous everywhere, but its derivative is unbounded and oscillates wildly in the neighborhood of zero.

(a) Note that \(f(x, y) = \tan(xy)\) is continuous as long as \(|xy| < \frac{\pi}{2}\). Beyond that, it is unbounded (and not continuous). This rules out global convergence, since that requires \(y\) to be unrestricted. Any pair of closed and bounded intervals that contains \((0, \frac{\pi}{2})\) and contains no points violating \(|xy| < \frac{\pi}{2}\) will suffice. Note that the derivative \(|f_y(x, y)| = |x| \sec^2(\tan(xy))\) will be bounded on such an interval, which ensures Lipschitz continuity.

(b) This function is continuous everywhere. The derivative, \(|f_y(x, y)| = |x\cos(xy)| \leq |x|\) is bounded for each fixed value of \(x\), which suffices to meet the Lipschitz requirement. Both theorems apply. Any closed rectangle (local) or strip (global) is fine.

(c) \(f(x, y) = \frac{xy}{x^2 + y^2}\) is not continuous at \((0, 0)\), so no rectangle containing \((0, 0)\) will work. Neither theorem applies. The lack of continuity may be observed by considering two paths to the origin, such as \(f(0, t) = 0\) and \(f(t, t) = \frac{1}{2}\).

(d) Note that \(g(x)\) is continuous at 0 (as can be shown by the squeeze theorem). Then, \(f(x, y) = g(x)y\) is continuous everywhere. Further, \(|f_y(x, y)| = |g(x)|\) is a suitable Lipschitz bound for all \(x\) and \(y\). Both theorems apply. Any closed rectangle (local) or strip (global) is fine.

(e) \(|f_y(x, y)| = |x|\sin(\frac{1}{y}) - \frac{1}{y^2}\cos(\frac{1}{y})|\) is not bounded as \(y \to 0\). The function is continuous everywhere, but it is not Lipschitz in \(y\) near \(y = 0\). Neither theorem applies.
Problem 3 (5 points)

The definition of an inner product on functions on \([a, b]\) presented in class can be generalized. Let \(\rho\) be a continuous, positive function on \([a, b]\). Then, I can define

\[
\langle f, g \rangle_{\rho} = \int_{a}^{b} \rho(x)f(x)g(x) \, dx.
\]

I can also define a norm from this, via \(\|f\|_{\rho} = \sqrt{\langle f, f \rangle_{\rho}}\). Show that this norm \(\| \cdot \|_{\rho}\) satisfies the identity \(\| f + g \|_{\rho} \leq \| f \|_{\rho} + \| g \|_{\rho}\).

Note that \(\langle f, g \rangle_{\rho} = \langle f\sqrt{\rho}, g\sqrt{\rho} \rangle\) and \(\|f\|_{\rho} = \|f\sqrt{\rho}\|\). Then,

\[
\begin{align*}
\| f + g \|_{\rho} &= \|(f + g)\sqrt{\rho}\|
\leq \| f \sqrt{\rho} \| + \| g \sqrt{\rho} \|
= \| f \|_{\rho} + \| g \|_{\rho}.
\end{align*}
\]
Problem 4 (5 points)

Let

\[ f(x) = \begin{cases} 
1 & |x| < \frac{\pi}{2} \\
0 & \frac{\pi}{2} \leq |x| \leq \pi 
\end{cases} \]

Compute the Fourier series for \( f(x) \).

Since \( f(x) \) is even, \( b_n = 0 \).

\[
a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx
= \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \, dx
= 1
\]

\[
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \quad n > 0
= \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos nx \, dx
= \frac{1}{n\pi} \left[ \sin nx \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}
= \frac{2\sin \frac{2\pi}{2}}{n\pi}
= \frac{a_{2n}}{n\pi}
\]

\[
a_{2n} = 0
\]

\[
a_{2n+1} = \frac{2(-1)^n}{(2n+1)\pi}
\]

\[
f(x) = \frac{1}{2} + \sum_{n=0}^{\infty} \frac{2(-1)^n \cos(2n+1)x}{(2n+1)\pi}
\]
Problem 5 (5 points)

Construct a sequence of functions $\theta_0, \theta_1, \theta_2, \theta_3$ with the following properties:

- The sequence is orthogonal on $[-1, 1]$.
- $\theta_k$ is a polynomial of degree $k$.
- $\theta_{2k}$ is an even function.
- $\theta_{2k+1}$ is an odd function.

From the last three requirements, we are looking for

$$
\theta_0 = 1 \quad \theta_1 = x \quad \theta_2 = x^2 + a \quad \theta_3 = x^3 + bx
$$

There is no normalization requirement, and many of the terms are omitted to make the functions even or odd. That reduces the work substantially. Note that the even and odd functions are automatically orthogonal, since integrating their product, an odd function, will always vanish on the symmetric interval $[-1, 1]$.

\[
\int_{-1}^{1} (1)(x^2 + a) \, dx = \left[ \frac{1}{3} x^3 + ax \right]_{-1}^{1} = \frac{2}{3} + 2a
\]

\[
a = -\frac{1}{3}
\]

\[
\int_{-1}^{1} (x)(x^3 + bx) \, dx = \left[ \frac{1}{5} x^5 + \frac{1}{3} bx^3 \right]_{-1}^{1} = \frac{2}{5} + \frac{2}{3} b
\]

\[
b = -\frac{3}{5}
\]

$$
\theta_0 = 1 \quad \theta_1 = x \quad \theta_2 = x^2 - \frac{1}{3} \quad \theta_3 = x^3 - \frac{3}{5} x
$$
Extra credit (5 points\(^\dagger\))

Construct a first order ODE for \(y(x)\), corresponding initial conditions \(y(x_0) = y_0\), and two distinct functions \(y_1(x)\) and \(y_2(x)\) such that:

- The global existence and uniqueness theorem applies.
- Both \(y_1\) and \(y_2\) are continuously differentiable on \((-\infty, \infty)\).
- Both \(y_1\) and \(y_2\) satisfy the ODE everywhere.
- Both \(y_1\) and \(y_2\) satisfy the initial conditions.

At first this task seems contradictory, but it really is not. The global existence and uniqueness theorem applies only in a closed and finite interval. Outside that interval, bad stuff may still happen. The functions \(y_1\) and \(y_2\) will necessarily agree identically around the initial conditions (because of uniqueness), but they can be defined piecewise. At this branch, we will need to violate both uniqueness theorems.

An easy way to go about this construction is to select two functions that are continuously differentiable everywhere and agree with each other in value in derivative at one point. I will choose \(g(x) = x^2\) and \(h(x) = -x^2\), where \(g(0) = g'(0) = h(0) = h'(0) = 0\). Next, I need to create an ODE which has both a solutions. Note that \(\frac{g'}{x} = \frac{h'}{x} = \frac{2}{x}\). Thus, I want to choose the ODE \(xy' = 2y\).

Note that the ODE satisfies the conditions of the global existence and uniqueness theorems everywhere except \(x = 0\). I will select \(x_0 = 2\) and \(y_0 = 4\). \(y_1(x) = x^2\) is one solution with suitable properties. By stitching together \(g\) and \(h\), I can construct a second solution: \(y_2(x) = x|x|\).

The global existence and uniqueness theorem applies in the interval \(x \in [1, 3]\) and \(y \in (-\infty, \infty)\).

\(^\dagger\)This exam is graded out of 25 points. Extra credit can be use to replace up to 5 points missed elsewhere on this exam, but the total grade for the exam cannot exceed 25 points.
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