Math 135-2, Homework 3

Solutions

Problem 53.4
Use the methods of both Examples 1 and 2 to solve each of the following differential equations:

(a) \( y'' + 5y' + 6y = 5e^{3t}, \ y(0) = y'(0) = 0. \)

(a) First, let’s use (13).

\[
L[A(t)] = \frac{1}{p(p^2 + 5p + 6)}
\]

\[
= \frac{1}{p(p + 3)(p + 2)}
\]

\[
= \frac{B}{p} + \frac{C}{p + 2} + \frac{D}{p + 3}
\]

\[
1 = B(p + 2)(p + 3) + C(p + 3) + D(p + 2)
\]

\[
1 = 6B \quad p = 0
\]

\[
1 = -2C \quad p = -2
\]

\[
1 = 3D \quad p = -3
\]

\[
L[A(t)] = \frac{1}{6p} - \frac{1}{2(p + 2)} + \frac{1}{3(p + 3)}
\]

\[
A(t) = \frac{1}{6} - \frac{1}{2}e^{-2t} + \frac{1}{3}e^{-3t}
\]

\[
f(t) = 5e^{3t}
\]

\[
f'(t) = 15e^{3t}
\]

\[
y(t) = \int_0^t A(t - \tau)f'(\tau)\,d\tau + f(0)A(t)
\]

\[
= \int_0^t \left( \frac{1}{6} - \frac{1}{2}e^{-2(t-\tau)} + \frac{1}{3}e^{-3(t-\tau)} \right) (15e^{3\tau})\,d\tau + 5\left( \frac{1}{6} - \frac{1}{2}e^{-2t} + \frac{1}{3}e^{-3t} \right)
\]

\[
= \int_0^t \frac{5}{2}e^{3\tau} - \frac{15}{2}e^{-2t+5\tau} + 5e^{-3t+6\tau}\,d\tau + \frac{5}{6} - \frac{5}{2}e^{-2t} + \frac{5}{3}e^{-3t}
\]

\[
= \left[ \frac{5}{6}e^{3\tau} - \frac{3}{2}e^{-2t+5\tau} + \frac{5}{6}e^{-3t+6\tau} \right]_0^t + \frac{5}{6} - \frac{5}{2}e^{-2t} + \frac{5}{3}e^{-3t}
\]

\[
= \frac{5}{6}(e^{3t} - e^{-2t}) + \frac{5}{6}(e^{3t} - e^{-3t}) + \frac{5}{6} - \frac{5}{2}e^{-2t} + \frac{5}{3}e^{-3t}
\]

\[
= \frac{1}{6}e^{3t} - e^{-2t} + \frac{5}{6}e^{-3t}
\]
Next, let’s repeat with (12).

\[ L[h(t)] = \frac{1}{p^2 + 5p + 6} = \frac{1}{(p + 3)(p + 2)} = \frac{C}{p + 2} + \frac{D}{p + 3} \]

1. \( C(p + 3) + D(p + 2) = 1 \)
2. \( C = p = -2 \)
3. \( D = p = -3 \)

\[ L[h(t)] = \frac{1}{p + 2} - \frac{1}{p + 3} \]

\( h(t) = e^{-2t} - e^{-3t} \)

\( f(t) = 5e^{3t} \)

\( y(t) = \int_0^t h(t - \tau)f(\tau) \, d\tau \)

\[ = \int_0^t \left( e^{-2(t-\tau)} - e^{-3(t-\tau)} \right)(5e^{3\tau}) \, d\tau \]

\[ = \int_0^t 5e^{-2t+5\tau} - 5e^{-3t+6\tau} \, d\tau \]

\[ = \left[ e^{-2t+5\tau} - \frac{5}{6} e^{-3t+6\tau} \right]_0^t \]

\[ = e^{3t} - e^{-2t} - \frac{5}{6} e^{3t} + \frac{5}{6} e^{-3t} \]

\[ = \frac{1}{6} e^{3t} - e^{-2t} + \frac{5}{6} e^{-3t} \]

**Problem 53.8**

The current \( I(t) \) in an electric circuit with inductance \( L \) and resistance \( R \) is given by the equation (4) in Section 13:

\[ L \frac{dI}{dt} + RI = E(t), \]

where \( E(t) \) is the impressed electromotive force. If \( I(0) = 0 \), use the methods of this section to find \( I(t) \) in each of the following cases:

(a) \( E(t) = E_0 u(t) \)
(b) \( E(t) = E_0 \delta(t) \)
(c) \( E(t) = E_0 \sin \omega t \)
\[
L[h(t)] = \frac{1}{Lp + R} = \frac{1}{Lp + R/L} \\
h(t) = \frac{1}{L} e^{-Rt/L} \\
I(t) = \int_0^t h(t - \tau)E(\tau) \, d\tau \\
= \int_0^t \frac{1}{L} e^{-R(t - \tau)/L} E(\tau) \, d\tau \\
= \frac{1}{L} e^{-Rt/L} \int_0^t e^{R\tau/L} E(\tau) \, d\tau \\
I(t) = \frac{1}{L} e^{-Rt/L} \int_0^t e^{R\tau/L} E_0 u(\tau) \, d\tau \quad \text{part (a)} \\
= \frac{E_0}{L} e^{-Rt/L} \int_0^t e^{R\tau/L} \, d\tau \\
= \frac{E_0}{L} e^{-Rt/L} \left[ \frac{1}{R} e^{R\tau/L} \right]_0^t \\
= \frac{E_0}{L} e^{-Rt/L} \left( \frac{1}{R} e^{R\tau/L} - \frac{L}{R} \right) \\
= \frac{E_0}{R} \left( 1 - e^{-Rt/L} \right) \\
I(t) = \frac{1}{L} e^{-Rt/L} \int_0^t e^{R\tau/L} E_0 \delta(\tau) \, d\tau \quad \text{part (b)} \\
= \frac{E_0}{L} e^{-Rt/L} \\
I(t) = \frac{1}{L} e^{-Rt/L} \int_0^t e^{R\tau/L} E_0 \sin \omega \tau \, d\tau \quad \text{part (c)} \\
= \frac{E_0}{L} e^{-Rt/L} \int_0^t e^{R\tau/L} \sin \omega \tau \, d\tau \\
= \frac{E_0}{L} e^{-Rt/L} \left( \int_0^t e^{(R/L + i\omega)\tau} \, d\tau \right) \\
= \frac{E_0}{L} e^{-Rt/L} \left( \left[ \frac{1}{R/L + i\omega} e^{(R/L + i\omega)\tau} \right]_0^t \right) \\
= \frac{E_0}{L} e^{-Rt/L} \left( \left( \frac{R/L - i\omega}{(R/L)^2 + \omega^2} e^{(R/L + i\omega)t} - 1 \right) \right) \\
= \frac{E_0}{L((R/L)^2 + \omega^2)} e^{-Rt/L} \left( (R/L - i\omega) \left( e^{Rt/L} \cos \omega t + i e^{Rt/L} \sin \omega t - 1 \right) \right) \\
= \frac{E_0}{L((R/L)^2 + \omega^2)} e^{-Rt/L} \left( R e^{Rt/L} \sin \omega t - \omega \left( e^{Rt/L} \cos \omega t - 1 \right) \right) \\
= \frac{E_0}{R^2 + L^2 \omega^2} \left( R \sin \omega t - L \omega \cos \omega t + L \omega e^{-Rt/L} \right)
Problem 69.2
Show that \( f(x, y) = y^{1/2} \)
(a) does not satisfy a Lipschitz condition on the rectangle \(|x| \leq 1\) and \(0 \leq y \leq 1\).
(b) does satisfy a Lipschitz condition on the rectangle \(|x| \leq 1\) and \(c \leq y \leq d\) where \(0 < c < d\).

(a) Let \( y_1 = 0 \) and \( y_2 = \varepsilon \).
\[
\frac{f(x, y_1) - f(x, y_2)}{y_1 - y_2} = \frac{\sqrt{y_1} - \sqrt{y_2}}{y_1 - y_2}
= \frac{1}{\sqrt{y_1} + \sqrt{y_2}}
= \frac{1}{\sqrt{\varepsilon}}
\]
which is unbounded.
(b) Noting \( y_1, y_2 \geq c > 0 \),
\[
\left| \frac{f(x, y_1) - f(x, y_2)}{y_1 - y_2} \right| = \left| \frac{\sqrt{y_1} - \sqrt{y_2}}{y_1 - y_2} \right|
= \frac{1}{\sqrt{y_1} + \sqrt{y_2}}
\leq \frac{1}{2\sqrt{c}}
= R
\]
provides a bound.

Problem 69.4
Show that \( f(x, y) = xy^2 \)
(a) satisfies a Lipschitz condition on the rectangle \(a \leq x \leq b\) and \(c \leq y \leq d\).
(b) does not satisfy a Lipschitz condition on any strip \(a \leq x \leq b\) and \(-\infty \leq y \leq \infty\).

(a) Note that \(|x| \leq \max(|a|, |b|) = A\) and \(|y| \leq \max(|c|, |d|) = C\).
\[
\left| \frac{f(x, y_1) - f(x, y_2)}{y_1 - y_2} \right| = \left| \frac{xy_1^2 - xy_2^2}{y_1 - y_2} \right|
= |x(y_1 + y_2)|
\leq |x|(|y_1| + |y_2|)
\leq 2AC
\]
is a bound.
(b) Choose any \( x \neq 0 \) (possible unless \( a = b \)), \( y_1 = 0 \), and \( y_2 \to \infty \).
\[
\left| \frac{f(x, y_1) - f(x, y_2)}{y_1 - y_2} \right| = \left| \frac{xy_1^2 - xy_2^2}{y_1 - y_2} \right|
= |x(y_1 + y_2)|
= |x|y_2
\to \infty
is unbounded.

**Problem A**

The problem $yy' = 1$, $y(0) = 0$ seems like it should have no solution. Show that it actually has two solutions. How is this possible? This demonstrates that plugging the initial conditions into an ODE and producing a contradiction does not suffice to show that there is no solution.

\[
\begin{align*}
yy' &= 1 \\
\frac{1}{2}y^2 &= x + c \\
y &= \pm \sqrt{2(x + c)} \\
0 &= y(0) = \pm \sqrt{2c} \\
y &= \pm \sqrt{2x}
\end{align*}
\]

The two solutions are $y = \sqrt{2x}$ and $y = -\sqrt{2x}$. Plugging $y = 0$ into $yy' = 1$ and deriving a contradiction implicitly assumes that $y'$ is finite, which it is not. In actuality, plugging in $y = 0$ produces $0 \cdot \infty$, which is indeterminate.

**Problem B**

Consider the ODE $x^3 y' = 2y$.

(a) Find all solutions if $y(0) = 0$.

(b) Find all solutions if $y(0) = 1$.

(a) First, lets find the general solution. The equation is separable.

\[
\begin{align*}
x^3y' &= 2y \\
y^{-1}y' &= 2x^{-3} \\
\ln |y| &= -x^{-2} + c_0 \\
y &= c_1 e^{-x^{-2}}
\end{align*}
\]

This satisfies is a solution for any $c_1$. We have not lost any solutions by dividing by zero, since $y = 0$ is captured by $c_1 = 0$.

(b) We also know that $y(0) = 1$ is not possible since we have already worked out the general solution. Note that it is not sufficient to plug $x = 0$ and $y = 1$ into the ODE to derive a contradiction, as demonstrated by Problem A.

**Problem C**

Find the Lipschitz constant (or show that it does not have one) for each of the following functions on the indicated interval. (The Lipschitz constant is a tight bound for the Lipschitz condition.)

(a) $\cos x \sin x$, $(-\infty, \infty)$

(b) $|\sin x|$, $(-\infty, \infty)$

Note that if $f(x)$ is differentiable on some interval $[a, b]$, then the Lipschitz constant $L$ for that interval
is obtained by looking at its derivative.

\[
\frac{f(a) - f(b)}{a-b} = f'(c) \quad \text{for some } c \in [a, b]
\]

Thus, any value of this fraction that can be obtained is also obtained by the derivative somewhere in the interval. What is more, the derivative is obtained in the limit \( a \to c \) and \( b \to c \), so

\[
L = \sup_{a \leq z \leq b} |f'(z)|
\]

(a) This one is differentiable. \( L = \max_x |f'(x)| = \max_x |\cos 2x| = 1. \)

(b) From \( |x| = |y + (x - y)| \leq |y| + |x - y| \) and \( |y| = |x + (y - x)| \leq |x| + |y - x| \) we deduce \( |x| - |y| \leq |x - y| \).

Then,

\[
\left| \frac{f(x) - f(y)}{x - y} \right| = \left| \frac{\sin x - \sin y}{x - y} \right|
\leq \sup_z |\cos z|
= 1
\]

Now, I need to show that \( L = 1 \) is tight. This follows from

\[
\lim_{\epsilon \to 0^+} \left| \frac{f(\epsilon) - f(0)}{\epsilon - 0} \right| = \lim_{\epsilon \to 0^+} \left| \frac{\sin \epsilon - \sin 0}{\epsilon - 0} \right|
= \lim_{\epsilon \to 0^+} \frac{\sin \epsilon}{\epsilon}
= 1
\]

### Problem D

Derive the time delay rule

\[
L[u(x-a)f(x-a)] = e^{-ap}F(p).
\]

For which choices \( a \) is this rule valid?
\[ L[u(x-a)f(x-a)] = \int_0^\infty e^{-px}u(x-a)f(x-a)\,dx \]
\[ = \int_{-a}^\infty e^{-pz}u(z)f(z)\,dz \quad x = z + a \]
\[ = \int_0^\infty e^{-pz}u(z)f(z)\,dz + \int_{-a}^0 e^{-pz}u(z)f(z)\,dz \]
\[ = \int_0^\infty e^{-pz}f(z)\,dz + \int_{-a}^0 e^{-pz}u(z)f(z)\,dz \]
\[ = e^{-ap}\int_0^\infty e^{-pz}f(z)\,dz + \int_{-a}^0 e^{-pz}u(z)f(z)\,dz \]
\[ = e^{-ap}F(p) + \int_{-a}^0 e^{-pz}u(z)f(z)\,dz \]

If \( a \geq 0 \), then the remaining integral is over negative values of \( z \), for which \( u(z) = 0 \). Thus, we will have the desired identity. If \( a < 0 \), then the remaining integral will in general be nonzero, since all three factors will generally be nonzero. Thus, the identity is true only for \( a \geq 0 \).