Math Review

CS 130

1. Points
   - Locations: \( P, Q, R \)

2. Vectors
   - Direction, magnitude
   - No location
   - Used to indicate quantities like change in position, velocity, force, etc.
   - \( \mathbf{u}, \mathbf{v}, \mathbf{w} \) or \( \vec{u}, \vec{v}, \vec{w} \)
   - Note that positions are very often represented with vectors (displacement from the origin).

   Vector operations
   \[
   \begin{pmatrix}
   a_1 \\
   a_2 \\
   a_3 
   \end{pmatrix} +
   \begin{pmatrix}
   b_1 \\
   b_2 \\
   b_3 
   \end{pmatrix} =
   \begin{pmatrix}
   a_1 + b_1 \\
   a_2 + b_2 \\
   a_3 + b_3 
   \end{pmatrix}
   \]
   \[
   \begin{pmatrix}
   a_1 \\
   a_2 \\
   a_3 
   \end{pmatrix} -
   \begin{pmatrix}
   b_1 \\
   b_2 \\
   b_3 
   \end{pmatrix} =
   \begin{pmatrix}
   a_1 - b_1 \\
   a_2 - b_2 \\
   a_3 - b_3 
   \end{pmatrix}
   \]
   \[k \begin{pmatrix}
   a_1 \\
   a_2 \\
   a_3 
   \end{pmatrix} =
   \begin{pmatrix}
   ka_1 \\
   ka_2 \\
   ka_3 
   \end{pmatrix}
   \]

   Coordinates; \( a_1, a_2, a_3 \) are coordinates, \( i, j, k \) are a basis
   \[
   \begin{pmatrix}
   a_1 \\
   a_2 \\
   a_3 
   \end{pmatrix} =
   a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}
   \]

   - In practice, the basis is normally understood (usually the standard one above), and only the coordinates are written explicitly.

3. Linear operator
   - Special type of function \( f(x) \)
   - Defining property: \( af(x) + bf(y) = f(ax + by) \)
   - Typical form \( f(x) = ax \), though not always. (Differentiation is a linear operator.)
   - For example, if \( x, y \) are vectors, then the linear operators are matrices. That is, \( f(\mathbf{u}) = \mathbf{M} \mathbf{u} \).
   - Note that \( f(0) = 0 \).

4. Affine operator
   - Typical form: \( f(x) = ax + b \).
   - Note that \( f(0) \) is not necessarily 0.
   - The definition is fairly general. \( x \) and \( b \) could be vectors; \( a \) could be a matrix.

5. Lines
   - \( L(t) = P + t \mathbf{u} \).
6. Line segment
   - Straight line connecting endpoints $P, Q$.
   - $L(t) = (1 - t)P + tQ; \ t \in [0, 1]$.
   - Note: $L(t) = (1 - t)P + tQ = P + t(Q - P), \ u = Q - P.$

7. Dot product
   - $\mathbf{u} \cdot \mathbf{v} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1b_1 + a_2b_2 + a_3b_3$
   - Well defined for any size of vector (any number of components), but both vectors must have the same size.
   - $\mathbf{u} \cdot \mathbf{u} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = a_1^2 + a_2^2 + a_3^2 = \|\mathbf{u}\|^2; \ ||\mathbf{u}\||$ is the length of the vector.
   - $\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}||\mathbf{v}\| \cos \theta; \ \theta$ is the angle between the vectors.
   - Sign tells you how “aligned” two vectors are:
     - $\mathbf{u} \cdot \mathbf{v} = 0$
     - $\mathbf{u} \cdot \mathbf{v} > 0$
     - $\mathbf{u} \cdot \mathbf{v} < 0$

8. Cross Product
   - Defined in 3D only.
     - $\mathbf{u} \times \mathbf{v} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$
     - $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$
     - (notation abuse)
     - Note that $\mathbf{w} = \mathbf{u} \times \mathbf{v}$ is a vector.
     - Length: $||\mathbf{w}|| = ||\mathbf{u} \times \mathbf{v}|| = ||\mathbf{u}||||\mathbf{v}|| \sin \theta.$
Direction: \( \mathbf{w} \) is orthogonal to \( \mathbf{u} \) and \( \mathbf{v} \): \( \mathbf{u} \cdot \mathbf{w} = \mathbf{v} \cdot \mathbf{w} = 0 \).

- Note that both \( \mathbf{w} \) and \( -\mathbf{w} \) have these properties; need a convention to disambiguate direction.
- How I remember it: \( \mathbf{i} \times \mathbf{j} = \mathbf{k} \).
- Others use “right hand rule” and similar devices.
- Note that \( \mathbf{u} \times \mathbf{u} = \mathbf{0} \), since \( \theta = 0 \) and \( \| \mathbf{u} \times \mathbf{u} \| = \| \mathbf{u} \| \| \mathbf{u} \| \sin \theta = 0 \).

9. Dot product and cross product have lots of useful properties, which are straightforward to check:

\[
\begin{align*}
\mathbf{u} \cdot \mathbf{v} &= \mathbf{v} \cdot \mathbf{u} \\
\mathbf{u} \times \mathbf{v} &= -\mathbf{v} \times \mathbf{u} \\
(\mathbf{a} \mathbf{u}) \cdot \mathbf{v} &= \mathbf{a} (\mathbf{u} \cdot \mathbf{v}) \\
(\mathbf{a} \mathbf{u}) \times \mathbf{v} &= \mathbf{a} (\mathbf{u} \times \mathbf{v}) \\
(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} &= (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} \\
\|\mathbf{u}\|^2 \|\mathbf{v}\|^2 &= (\mathbf{u} \cdot \mathbf{v})^2 + \|\mathbf{u} \times \mathbf{v}\|^2
\end{align*}
\]

10. Matrices

- Table of numbers

\[
\mathbf{M} = \begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{pmatrix} \quad \mathbf{N} = \begin{pmatrix}
b_{11} & b_{12} \\
b_{21} & b_{22} \\
b_{31} & b_{32} \\
b_{41} & b_{42}
\end{pmatrix}
\]

- Entries are indexed \( a_{ij} \); row is \( i \), column is \( j \).
- Matrices have dimensions that indicate their size and shape: \( \mathbf{M} \) is \( 3 \times 3 \), \( \mathbf{N} \) is \( 4 \times 2 \).
- Matrix-vector multiply:

\[
\begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_2 \\
u_3
\end{pmatrix} = \begin{pmatrix}
a_{11}u_1 + a_{12}u_2 + a_{13}u_3 \\
a_{21}u_1 + a_{22}u_2 + a_{23}u_3 \\
a_{31}u_1 + a_{32}u_2 + a_{33}u_3
\end{pmatrix}
\]

- Pattern: \( \mathbf{v} = \mathbf{M} \mathbf{u} \) is calculated as \( v_i = \sum_j m_{ij} u_j \).
- Can be thought of as a linear operator: \( \mathbf{v} = f(\mathbf{u}) = \mathbf{M} \mathbf{u} \).
- Composition: \( f(\mathbf{u}) = \mathbf{M} \mathbf{u} \), \( g(\mathbf{u}) = \mathbf{P} \mathbf{u} \), \( h(\mathbf{u}) = f(g(\mathbf{u})) \) or \( h = f \circ g \).

\[
\mathbf{w} = f(g(\mathbf{u})) = f(\mathbf{P} \mathbf{u}) = \mathbf{M}(\mathbf{P} \mathbf{u}) = (\mathbf{MP})\mathbf{u} = \mathbf{Q}\mathbf{u}
\]

\[
w_i = \sum_j m_{ij} \left( \sum_k p_{jk} u_k \right) = \sum_k \left( \sum_j m_{ij} p_{jk} \right) u_k
\]

\[
\mathbf{Q} = \mathbf{MP} \iff q_{ik} = \sum_j m_{ij} p_{jk}
\]

- This is the rule for matrix multiplication.
- Transpose: \( \mathbf{N} = \mathbf{M}^T \) is defined by \( n_{ij} = m_{ji} \).