CS 130, Midterm

Solutions

1	2	3	4	5	6	7	8	9	10

Read the entire exam before beginning. **Manage your time carefully.** This exam has 26 points; you need 20 to get full credit. Additional points are extra credit.

Problem 1 (2 points)

What is the angle between the vectors $\langle 1, 2 \rangle$ and $\langle 3, 4 \rangle$? (You do not need to simplify your answer.)

Using $u \cdot v = \|u\| \|v\| \cos \theta$ we have

$$\theta = \cos^{-1}\left(\frac{(1)(3) + (2)(4)}{\sqrt{1^2 + 2^2}\sqrt{3^2 + 4^2}}\right)$$
$$\theta = \cos^{-1}\left(\frac{11}{5\sqrt{5}}\right)$$

Problem 2 (2 points)

What might you see if you look at the display on your cell phone under a magnifying glass? (Assuming the phone is displaying a white background.)

You would see a pattern of red, green, and blue lights.

Problem 3 (2 points)

Given a 2D vector $\mathbf{u} = \langle x, y \rangle$, construct a vector $\mathbf{v}$ that is orthogonal to $\mathbf{u}$ but has the same length as $\mathbf{u}$.

Let $\mathbf{v} = \langle a, b \rangle$. Since they are orthogonal, $ax + by = 0$, so $b = -\frac{ax}{y}$.

$$\|y\|^2 = a^2 + \left(-\frac{ax}{y}\right)^2 = \frac{a^2}{y^2}(x^2 + y^2) = \|x\|^2 = x^2 + y^2$$

Thus we must have $a = \pm y$ (both work ok). Let $a = -y$ so that $b = x$. Thus, $\mathbf{v} = \langle -y, x \rangle$. (You could also obtain this result by applying a 2D rotation by angle $\frac{\pi}{2}$. Rotations are not on the coverage list for this exam, but if you know them, you are always welcome to use them.)

Problem 4 (2 points)

Given a ray (endpoint P and direction $\mathbf{u}$) and a segment (endpoints A and B) **in 2D**, (1) determine whether they intersect and if so (2) find the intersection location. (You may use the results of the previous problem, even if you have not solved it.)

Let v be a vector orthogonal to u and w be a vector orthogonal to $B - A$. (These are constructed in the previous problem).

$$\begin{aligned}
x &= P + tu \\
x &= A + \lambda(B - A) \\
P - A &= \lambda(B - A) - tu \\
v \cdot (P - A) &= \lambda v \cdot (B - A) \\
\lambda &= \frac{v \cdot (P - A)}{v \cdot (B - A)} \\
w \cdot (P - A) &= -tw \cdot u \\
t &= -\frac{w \cdot (P - A)}{w \cdot u}
\end{aligned}$$

There is an intersection if and only if $t \geq 0$ and $0 \leq \lambda \leq 1$.

Problem 5 (2 points)

Find the normal direction for the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$

$$N = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$n = \frac{N}{\|N\|} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Problem 6 (2 points)

When we look into a tank of water, we observe distortion due to refraction. For example, a pencil dipped into the water will appear bent. When we look through the lenses of a pair of glasses, what we see on the other side appears distorted. However, when we look through a window, we do not observe appreciable distortion. Why is significant distortion observed in the first two cases but not the third?

The real mystery here is why the third case does not show appreciable distortion. The light travels through two surfaces - once entering the glass and once leaving it. Let θ_0 be the angle of the light incoming, θ_1 be the angle in the glass, and θ_2 be the angle of the light leaving the glass. If $n_0 = n_2 = 1$ and $n_1 > 1$, then by Snell's law $n_0 \sin\theta_0 = n_1 \sin\theta_1 = n_2 \sin\theta_2$, so that $\theta_0 = \theta_2$. Note that this only works because the two surfaces are parallel. The light effectively passes through the glass without bending, so very little distortion is observed. In the other two cases, the outgoing light is at a different angle from the incoming light, so distortion is observed. In the case of the lens, there are two surfaces as in the window case, but they are not parallel.

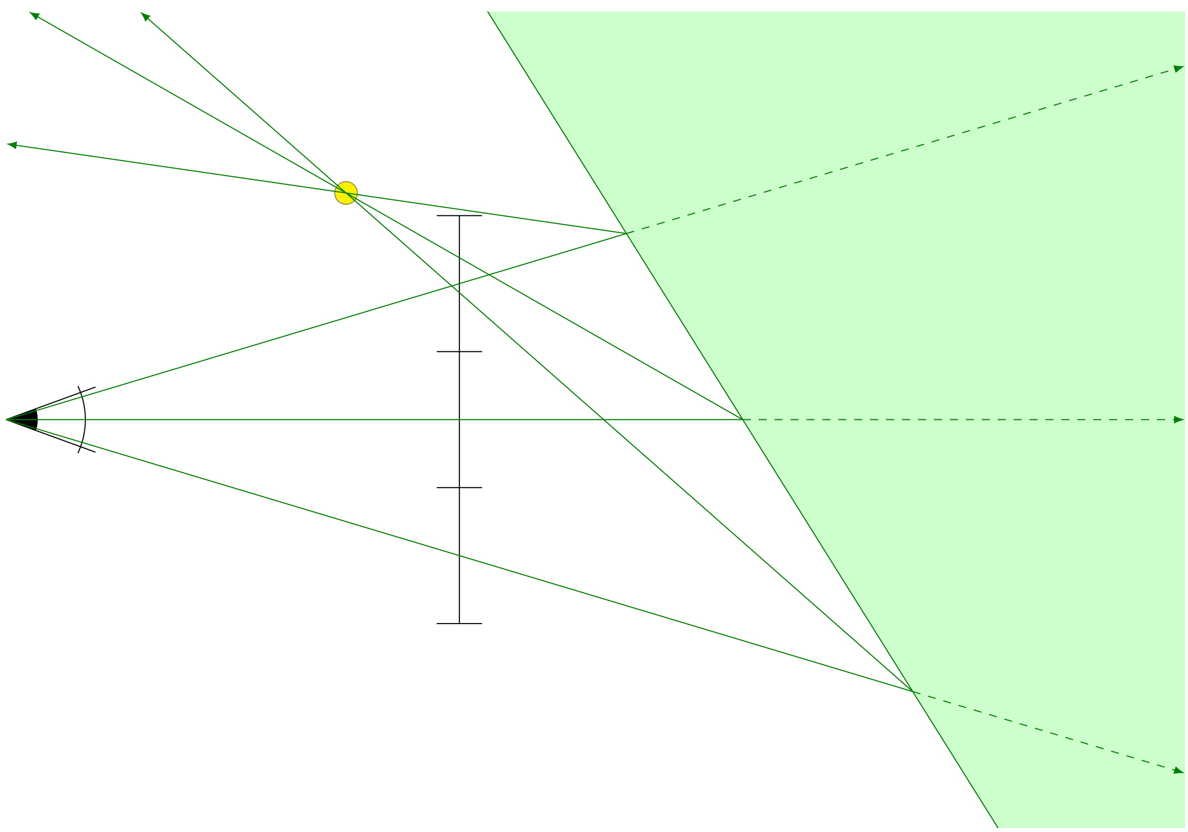
Problem 7 (2 points)

Construct a one-to-one function $f(x)$ mapping from $[a, b]$ to $[c, d]$.

$$
\begin{aligned}
f(x) &= rx + s \\
c &= ra + s \\
d &= rb + s \\
(d - c) &= r(b - a) \\
r &= \frac{d - c}{b - a} \\
s &= c - ra \\
s &= \frac{cb - ad}{b - a} \\
f(x) &= \frac{d - c}{b - a}x + \frac{cb - ad}{b - a}
\end{aligned}
$$

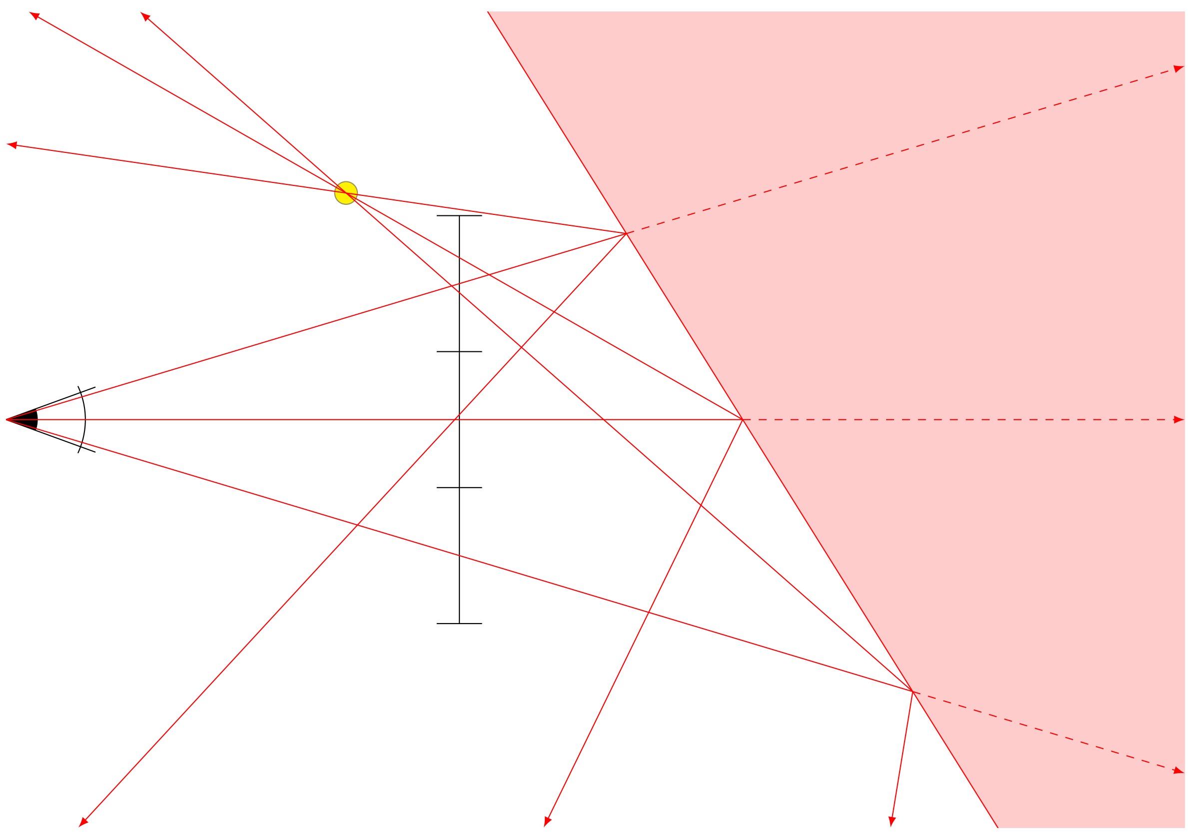
Problem 8 (4 points)

Below is a simple 2D raytracing setup. The 1D image has three pixels. The green object is made of wood. The yellow circle is a point light. Draw all of the rays that would be cast while raytracing this scene. (Tip: use the edge of a piece of paper or a pencil to draw rays; it helps.)



Problem 9 (4 points)

Below is a simple 2D raytracing setup. The 1D image has three pixels. The red object is reflective. The yellow circle is a point light. Draw all of the rays that would be cast while raytracing this scene. (Tip: use the edge of a piece of paper or a pencil to draw rays; it helps.)



Problem 10 (4 points)

Below is a simple 2D raytracing setup. The 1D image has three pixels. The blue object is made from glass. The yellow circle is a point light. Draw all of the rays that would be cast while raytracing this scene. (Tip: use the edge of a piece of paper or a pencil to draw rays; it helps.)