Math Review

CS 130

1. Points
   - Locations: \( P, Q, R \)

2. Vectors
   - Direction, magnitude
   - No location
   - Used to indicate quantities like change in position, velocity, force, etc.
   - \( u, v, w \) or \( \vec{u}, \vec{v}, \vec{w} \)
   - Note that positions are very often represented with vectors (displacement from the origin).
   - Vector operations
     \[
     \begin{pmatrix}
     a_1 \\
     a_2 \\
     a_3 
     \end{pmatrix}
     +
     \begin{pmatrix}
     b_1 \\
     b_2 \\
     b_3 
     \end{pmatrix}
     =
     \begin{pmatrix}
     a_1 + b_1 \\
     a_2 + b_2 \\
     a_3 + b_3 
     \end{pmatrix}
     \]
     \[
     \begin{pmatrix}
     a_1 \\
     a_2 \\
     a_3 
     \end{pmatrix}
     -
     \begin{pmatrix}
     b_1 \\
     b_2 \\
     b_3 
     \end{pmatrix}
     =
     \begin{pmatrix}
     a_1 - b_1 \\
     a_2 - b_2 \\
     a_3 - b_3 
     \end{pmatrix}
     \]
     \[
     k
     \begin{pmatrix}
     a_1 \\
     a_2 \\
     a_3 
     \end{pmatrix}
     =
     \begin{pmatrix}
     ka_1 \\
     ka_2 \\
     ka_3 
     \end{pmatrix}
     \]
   - Coordinates; \( a_1, a_2, a_3 \) are coordinates, \( i, j, k \) are a basis
     \[
     \begin{pmatrix}
     a_1 \\
     a_2 \\
     a_3 
     \end{pmatrix}
     =
     a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}
     \]
   - In practice, the basis is normally understood (usually the standard one above), and only the coordinates are written explicitly.

3. Linear operator
   - Special type of function \( f(x) \)
   - Defining property: \( af(x) + bf(y) = f(ax + by) \)
   - Typical form \( f(x) = ax \), though not always. (Differentiation is a linear operator.)
   - For example, if \( x, y \) are vectors, then the linear operators are matrices. That is, \( f(u) = Mu \).
   - Note that \( f(0) = 0 \).

4. Affine operator
   - Typical form: \( f(x) = ax + b \).
   - Note that \( f(0) \) is not necessarily 0.
   - The definition is fairly general. \( x \) and \( b \) could be vectors; \( a \) could be a matrix.

5. Lines
   - \( L(t) = P + tu \).
• Point on the line: \( P \)
• Direction of the line: \( u \)
• \( t \) is a parameter that tells where a point is along the line.

6. Line segment
• Straight line connecting endpoints \( P, Q \).
• \( L(t) = (1 - t)P + tQ; \ t \in [0, 1] \).
• Note: \( L(t) = (1 - t)P + t(Q - P), \ u = Q - P \).

7. Dot product
• \( u \cdot v = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1b_1 + a_2b_2 + a_3b_3 \)
• Well defined for any size of vector (any number of components), but both vectors must have the same size.
• \( u \cdot u = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = a_1^2 + a_2^2 + a_3^2 = \|u\|^2; \ |u| \) is the length of the vector.
• \( u \cdot v = \|u\||v||\cos \theta \); \( \theta \) is the angle between the vectors.
• Sign tells you how “aligned” two vectors are:

\[
\begin{align*}
\mathbf{u} \cdot \mathbf{v} &= 0 & \mathbf{u} \cdot \mathbf{v} &> 0 \\
\mathbf{u} \cdot \mathbf{v} &< 0
\end{align*}
\]

8. Cross Product
• Defined in 3D only.
\[
\mathbf{u} \times \mathbf{v} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}
\]
\[
= \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)i + (a_3b_1 - a_1b_3)j + (a_1b_2 - a_2b_1)k
\]
(notation abuse)
• Note that \( \mathbf{w} = \mathbf{u} \times \mathbf{v} \) is a vector.
• Length: \( \|\mathbf{w}\| = \|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\| \sin \theta \).
9. Dot product and cross product have lots of useful properties, which are straightforward to check:

\[
\begin{align*}
\mathbf{u} \cdot \mathbf{v} &= \mathbf{v} \cdot \mathbf{u} \\
(a\mathbf{u}) \cdot \mathbf{v} &= a(\mathbf{u} \cdot \mathbf{v}) \\
\mathbf{u} \times \mathbf{v} &= -\mathbf{v} \times \mathbf{u} \\
(a\mathbf{u}) \times \mathbf{v} &= a(\mathbf{u} \times \mathbf{v}) \\
\mathbf{u} + \mathbf{w} &= \mathbf{u} \cdot \mathbf{v} + \mathbf{w} \cdot \mathbf{v} \\
\mathbf{u} \times \mathbf{v} + \mathbf{w} \times \mathbf{v} &= (\mathbf{u} + \mathbf{w}) \times \mathbf{v} \\
\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) &= \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) \\
\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) &= (\mathbf{u} \cdot \mathbf{w}) \mathbf{v} - (\mathbf{u} \cdot \mathbf{v}) \mathbf{w} \\
\|\mathbf{u}\|\|\mathbf{v}\|^2 &= (\mathbf{u} \cdot \mathbf{v})^2 + \|\mathbf{u} \times \mathbf{v}\|^2
\end{align*}
\]

10. Matrices

- Table of numbers

\[
\mathbf{M} = \begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{pmatrix} \quad \mathbf{N} = \begin{pmatrix}
b_{11} & b_{12} \\
b_{21} & b_{22} \\
b_{31} & b_{32} \\
b_{41} & b_{42}
\end{pmatrix}
\]

- Entries are indexed \(a_{ij}\); row is \(i\), column is \(j\).

- Matrices have dimensions that indicate their size and shape: \(\mathbf{M}\) is \(3 \times 3\), \(\mathbf{N}\) is \(4 \times 2\).

- Matrix-vector multiply:

\[
\begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{pmatrix}
\begin{pmatrix}
u_1 \\
v_2 \\
v_3
\end{pmatrix} = \begin{pmatrix}
(a_{11}u_1 + a_{12}u_2 + a_{13}u_3) \\
(a_{21}u_1 + a_{22}u_2 + a_{23}u_3) \\
(a_{31}u_1 + a_{32}u_2 + a_{33}u_3)
\end{pmatrix}
\]

- Pattern: \(\mathbf{v} = \mathbf{M}\mathbf{u}\) is calculated as \(v_i = \sum_j m_{ij}u_j\)

- Can be thought of as a linear operator: \(\mathbf{v} = f(\mathbf{u}) = \mathbf{M}\mathbf{u}\).

- Composition: \(f(\mathbf{u}) = \mathbf{M}\mathbf{u}\), \(g(\mathbf{u}) = \mathbf{P}\mathbf{u}\), \(h(\mathbf{u}) = f(g(\mathbf{u}))\) or \(h = f \circ g\).

\[
\mathbf{w} = f(g(\mathbf{u})) = f(\mathbf{P}\mathbf{u}) = \mathbf{M}(\mathbf{P}\mathbf{u}) = (\mathbf{MP})\mathbf{u} = \mathbf{Q}\mathbf{u}
\]

\[
w_i = \sum_j m_{ij}\left(\sum_k p_{jk}u_k\right) = \sum_k \left(\sum_j m_{ij}p_{jk}\right)u_k
\]

\[
\mathbf{Q} = \mathbf{MP} \iff q_{ik} = \sum_j m_{ij}p_{jk}
\]

- This is the rule for matrix multiplication.

- Transpose: \(\mathbf{N} = \mathbf{M}^T\) is defined by \(n_{ij} = m_{ji}\)